

The balancedness of N -complexes based on cotorsion pairs

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Abstract — Let Φ be an abelian category, and (A, B) be a complete hereditary cotorsion pair on Φ . We define relative cohomology groups based on (A, B) in the category of N -complexes on Φ . Especially we are devoted to consider the balancedness of the relative cohomology functors.

Keywords— N -complex; cotorsion pair; relative cohomology group; balancedness

MSC(2010): 81E10; 18G25; 18G35

I. INTRODUCTION

The notion of N -complexes was introduced by Mayer [15] in his study of simplicial complexes. In 1996, Kapranov [10] and Dubois-Violette [5] gave an abstract framework of homological theory of N -complexes. Since then the notion of N -complexes has concerned by many authors, for example [7,9,10,20], and some important applications to theoretical physics, quantum theories and representation theories of algebras have been found. In 2015, Yang and Ding [20] provided an effective construction of left and right triangles, and showed that both the homotopy category and the derived category of N -complexes are pretriangulated categories. Later Iyama, Kato and Miyachi [9] further showed that the homotopy category and derived category of N -complexes are triangulated categories. Recently, Luo and Zhu [14] described how complete hereditary cotorsion pairs gave rise to model structures on the derived category of N -complexes.

The relative cohomology theory in various abelian categories was studied by many authors, for example [1,2,8,11,17]. Furthermore, the balancedness of the relative cohomology functors is one of the most important studies in the relative cohomology theory. Let R is an arbitrary ring. In 2004, Holm [8] defined relative cohomology groups in module category and showed that if R -module M , with a proper Gorenstein projective

resolution, has finite Gorenstein projective dimension and R -module N , with a proper Gorenstein injective resolution, has finite Gorenstein injective dimension, then there exist isomorphisms

$$\text{Ext}_{GP}^n(M, N) = \text{Ext}_{GI}^n(M, N) \quad \text{for } n \in \mathbb{Z} \quad (\text{see [8], theorem 3.6}).$$

In complex category, Liu [11] defined relative cohomology groups and the balancedness was already considered by showed that if X is a complex with finite Gorenstein projective dimension and Y is a complex with finite Gorenstein injective dimension, then for each $n \in \mathbb{Z}$ there exists an isomorphism (*):

$$\text{Ext}_{GP}^n(X, Y) = \text{Ext}_{GI}^n(X, Y), \quad \text{where } X \text{ has a strict Gorenstein projective precover and } Y \text{ has a strict Gorenstein injective preenvelope.}$$

Recently, Lu and Di [13] extended the (*) to the category of N -complexes.

Cotorsion pairs (or cotorsion theories) were introduced by Salce [16] and its importance in homological algebra has been showed by its use in the proof of the flat cover conjecture in [3]. Liu and Xie [12] considered relative cohomology theories based on cotorsion pairs in the setting of unbounded complexes of modules over an associative ring R . In [12], suppose (A, L) and (L, B) be two complete hereditary cotorsion pairs in the R -Mod, then by [6] and [18], $(dg\tilde{A}, \tilde{L})$ and $(\tilde{L}, dg\tilde{B})$ be two complete hereditary cotorsion pairs in the category of R -complexes. For any complex X, Y and any $n \in \mathbb{Z}$, Liu and Xie defined the n th relative cohomology groups $\text{Ext}_{dg\tilde{A}}^n(X, Y)$ as $H^n \text{Hom}(A, Y)$ and $\text{Ext}_{dg\tilde{B}}^n(X, Y)$ as $H^n \text{Hom}(X, B)$, where $A \rightarrow X$ is a special $dg\tilde{A}$ -precover of X and $Y \rightarrow B$ is a

special $dg\tilde{B}$ -preenvelope of Y . Of course, Liu and Xie also discussed the balancedness that there are an isomorphism $Ext_{dg\tilde{A}}^n(X, Y) = Ext_{dg\tilde{B}}^n(X, Y)$.

Motivated by the above works of authors, we define in this paper relative cohomology groups based on cotorsion pair (A, B) in the category of N -complexes on Φ . Since Yang and Cao [19], for a cotorsion pair (χ, γ) in Φ , defined cotorsion pairs $(\tilde{\chi}_N, dg\tilde{\gamma}_N)$ and $(dg\tilde{\chi}_N, \tilde{\gamma}_N)$ in the category of N -complexes on Φ . So we suppose (A, L) be a complete hereditary cotorsion pair in Φ . Then, by Yang and Cao [19], $(dg\tilde{A}_N, \tilde{L}_N)$ is a complete hereditary cotorsion pair in the category of N -complexes on Φ . For any N -complexes X and Y any $n \in \mathbb{Z}$, we define the n th relative cohomology group $Ext_{dg\tilde{A}_N}^n(X, Y)$ as $H_n^t Hom_{\Phi}(A, Y)$, where $A \rightarrow Y$ is a special $dg\tilde{A}_N$ -precover of X . If (L, B) is a complete hereditary cotorsion pair in Φ , then the n th relative cohomology group $Ext_{dg\tilde{B}_N}^n(X, Y)$ is defined by a dual argument. For relative cohomology groups we show that there exists an isomorphism $Ext_{dg\tilde{A}_N}^n(X, Y) = Ext_{dg\tilde{B}_N}^n(X, Y)$.

II. PRELIMINARIES

By an N -complex $X (N \geq 2)$ we mean a sequence of objects in Φ

$$\cdots \xrightarrow{d} X_{n+1} \xrightarrow{d} X_n \xrightarrow{d} X_{n-1} \xrightarrow{d} \cdots$$

satisfying $d^N = 0$, that is, composing any N -consecutive morphisms gives 0. So a 2-complex is a chain complex in the usual sense. Let C and D be N -complexes. We denote by $Hom_{\Phi}(C, D)$ the sequence of abelian groups with $Hom_{\Phi}(C, D)_n = \prod_{t \in \mathbb{Z}} Hom_{\Phi}(C_t, D_{n+t})$ and such that if $f \in Hom_{\Phi}(C, D)_n$ then

$(d_n(f))_m = d_D f_m - (q)^n f_{m+1} d_C$, where q is a N th root of unity, $q^N = 1$ and $q \neq 1$. Then $Hom_{\Phi}(C, D)$ is an N -complex of abelian groups. f is called a chain

map of degree n if $d_n(f) = 0$. A chain map of degree 0 is called a morphism. We use $Hom_{C_N(\Phi)}(C, D)$ to denote the abelian group of morphisms from C to D and $Ext_{C_N(\Phi)}^i(C, D)$ for $i \geq 0$ to denote the groups we get from the right derived functor of Hom (see [10]).

In this way, we get a category of N -complexes, denoted by $C_N(\Phi)$, whose objects are N -complexes and whose morphisms are chain maps of degree 0. It is an abelian category. For an N -complexes X , there are $N-1$ choices for homology. For $t = 1, \dots, N$, we define

$$Z_n^t(X) = Ker(d_{n-(t-1)} \cdots d_{n-1} d_n) \text{ and}$$

$$B_n^t(X) = Im(d_{n+1} d_{n+2} \cdots d_{n+t}).$$

$$Z_n^1(X) = Ker d_n, \quad Z_n^N(X) = X_n, \quad B_n^1(X) = Im d_{n+1}$$

$$\text{and } B_n^N(X) = 0. \text{ Set } C_n^t(X) = X_n / B_n^t(X). \text{ Define}$$

$$H_n^t(X) = Z_n^t(X) / B_n^{N-t}(X) \text{ the amplitude homology}$$

objects of X for all t . We say that X is N -exact, or just exact, if $H_n^t(X) = 0$ for all integers n and t .

Two morphisms $f, g : X \rightarrow Y$ of N -complexes are called homotopic if there exists a collection of morphisms $\{s_n : X_n \rightarrow Y_{n+N-1}\}$ such that

$$g_n - f_n = d^{N-1} s_n + d^{N-2} s_{n-1} d + \cdots + s_{n-(N-1)} d^{N-1}$$

$$= \sum_{i=0}^{N-1} d^{(N-1)-i} s_{n-i} d^i, \quad \forall n.$$

If f and g are homotopic, then we write $f \sim g$.

We call a chain map f null homotopic if $f \sim 0$.

If Φ is an abelian category and \mathbf{F} a class of objects of Φ , then an \mathbf{F} -precover of an object X of Φ is a morphism $\phi : F \rightarrow X$, where $F \in \mathbf{F}$ and such that

$$Hom(G, F) \rightarrow Hom(G, X) \text{ is surjective for all } G \in \mathbf{F}.$$

The dual notions are that of an \mathbf{F} -preenvelope. An

\mathbf{F} -precover $\phi : F \rightarrow X$ is said to be a special

\mathbf{F} -precover if $Ext^1(G, Ker \phi) = 0$ for all $G \in \mathbf{F}$. An

F -preenvelope $\varphi: X \rightarrow F$ is special if

$Ext^1(Coker \varphi, G) = 0$ for all $G \in \mathcal{A}$. Let H a class of objects in Φ . We will denote the class of objects X satisfying $Ext^1(H', X) = 0$ respectively,

$Ext^1(X, H') = 0$ for every $H' \in H$ by H^\perp respectively, ${}^\perp H$. A pair of classes of objects (A, B) is said to be a cotorsion pair if $A^\perp = B$ and ${}^\perp B = A$. A cotorsion pair (A, B) is said to be complete if for any X there are exact sequence

$$0 \rightarrow X \rightarrow B_1 \rightarrow A_1 \rightarrow 0, 0 \rightarrow B_2 \rightarrow A_2 \rightarrow X \rightarrow 0$$

with $A_1, A_2 \in A$ and $B_1, B_2 \in B$. A cotorsion pair

(A, B) is said to be hereditary if whenever

$$0 \rightarrow A_2 \rightarrow A_1 \rightarrow A_3 \rightarrow 0$$

is exact with $A_1, A_3 \in A$ then A_2 is also in A , or equivalently, if

$$0 \rightarrow B_2 \rightarrow B_1 \rightarrow B_3 \rightarrow 0$$

is exact with $B_1, B_2 \in B$ then B_3 is also in B .

III. RELATIVE COHOMOLOGY GROUP

The following definition appeared in [19, Definition 3.1].

Definition 3.1 Let (A, B) be a cotorsion pair in Φ and X be an objective $C_N(\Phi)$.

(1) X is called an A N -complex if it is N -exact and $Z'_n(X) \in A$ for all n and $t = 1, \dots, N-1$.

(2) X is called a B N -complex if it is N -exact and for all n and $t = 1, \dots, N-1$.

(3) X is called a dgA N -complex if $X_n \in A$ for all n and every chain map $f: X \rightarrow Y$ is null homotopic where Y is a B N -complex.

(4) X is called a dgB N -complex if $X_n \in B$ for all n and every chain map $f: Y \rightarrow X$ is null homotopic where Y is an A N -complex.

We denote the class of A N -complex by \tilde{A}_N and the class of dgA N -complex by $dg\tilde{A}_N$. Similarly, the class of B N -complex by \tilde{B}_N and the class of

dgB N -complex by $dg\tilde{B}_N$. It follows from [19, Theorem 3.7] that if (A, B) is a complete hereditary cotorsion pair in Φ , then $(dg\tilde{A}_N, \tilde{B}_N)$ and $(\tilde{A}_N, dg\tilde{B}_N)$ are complete hereditary cotorsion pair in $C_N(\Phi)$.

In the following (A, L) will always stand for a complete hereditary cotorsion pair in Φ . Then $(dg\tilde{A}_N, \tilde{L}_N)$ is a complete hereditary cotorsion pairs in $C_N(\Phi)$. For any N -complex X there is an exact sequence of N -complexes

$$0 \rightarrow L \rightarrow A \rightarrow X \rightarrow 0$$

with $A \in dg\tilde{A}_N$ and $L \in \tilde{L}_N$. It is easy to see that $A \rightarrow X$ is a special $dg\tilde{A}_N$ -precover of X .

Lemma 3.2 Let (A, L) be a complete hereditary cotorsion pair in Φ , and X and X' be an N -complexes. Then, for each morphism $\alpha: X \rightarrow X'$ there exists a unique, up to homotopy, morphism $\beta: A \rightarrow A'$ such that the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & L & \rightarrow & A & \xrightarrow{\phi} & X \rightarrow 0 \\ & & & & \beta \downarrow & & \alpha \downarrow \\ 0 & \rightarrow & L' & \rightarrow & A' & \xrightarrow{\phi'} & X' \rightarrow 0 \end{array}$$

commutes, where $A, A' \in dg\tilde{A}_N$ and $L, L' \in \tilde{L}_N$ are such that the rows are exact.

Proof. Clearly there exists a morphism $\beta: A \rightarrow A'$ such that the diagram commutes since $A' \rightarrow X'$ is a $dg\tilde{A}_N$ -precover of X' . Suppose that $\beta': A \rightarrow A'$ is another one such that the diagram commutes. Then $\phi'(\beta - \beta') = 0$ and so $\beta - \beta': A \rightarrow L'$. By the completeness of cotorsion pair $(dg\tilde{A}_N, \tilde{L}_N)$, there is an exact sequence of N -complexes

$$0 \rightarrow A \xrightarrow{\tau} L_1 \rightarrow A_1 \rightarrow 0 \quad (**)$$

with $A_1 \in dg\tilde{A}_N$ and $L_1 \in \tilde{L}_N$. Note that $Ext^1(A_1, L') = 0$ and, so the sequence

$$Hom_{\Phi}(L_1, L') \rightarrow Hom_{\Phi}(A, L') \rightarrow 0$$

is exact. Hence there exists a morphism $\gamma: L_1 \rightarrow L'$ such that $\beta - \beta' = \gamma\tau$. Form (***) it follows $L_1 \in dg\tilde{A}_N$ since A and A' belong to $dg\tilde{A}_N$. Note that $L' \in \tilde{L}_N$. Thus the complex $Hom_{\Phi}(L_1, L')$ is exact, which implies that γ is homotopic to zero and therefore $\beta - \beta'$ is homotopic to zero.

According to lemma 3.2 we have the following definition.

Definition 3.3 Let X and Y be N -complexes. For each $n \in Z$, the n th relative cohomology group $Ext_{dg\tilde{A}_N}^n(X, Y)$, based on a cotorsion pair (A, B) , is defined by the equality

$$Ext_{dg\tilde{A}_N}^n(X, Y) = H_n^t H \circ \eta_{\Phi}(A, Y),$$

where $A \rightarrow X$ is a special $dg\tilde{A}_N$ -precover of X and $t = 1, \dots, N-1$.

Proposition 3.4 For each $n \in Z$, the assignment $(X, Y) \mapsto Ext_{dg\tilde{A}_N}^n(X, Y)$ defines a functor

$$C_N(\Phi) \times C_N(\Phi) \rightarrow Z-Mod$$

Proof It follows from lemma 3.2

Dual argument 3.5 Let (L, B) be a complete hereditary cotorsion pair in Φ . Then $(\tilde{L}_N, dg\tilde{B}_N)$ is a complete hereditary cotorsion pair in $C_N(\Phi)$. For any N -complex Y there is an exact sequence of N -complex

$$0 \rightarrow Y \rightarrow B \rightarrow H \rightarrow 0$$

With $B \in dg\tilde{B}_N$ and $H \in \tilde{L}_N$. It is easy to see that $Y \rightarrow B$ is a special $dg\tilde{B}_N$ -preenvelope of Y . For any N -complex and any $n \in Z, t = 1, \dots, N-1$. the n th relative cohomology group $Ext_{dg\tilde{B}_N}^n(X, Y)$ is defined by the equality

$$Ext_{dg\tilde{B}_N}^n(X, Y) = H_n^t H \circ \eta_{\Phi}(X, B).$$

By the following lemma, the n th relative cohomology group is well defined.

Lemma 3.6 Let (L, B) be a complete hereditary

cotorsion pair in Φ , Y and Y' and be an N -complexes. Then, for each morphism $\alpha: Y \rightarrow Y'$, there exists a unique, up to homotopy, morphism $\beta: B \rightarrow B'$ such that the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & Y & \longrightarrow & B & \rightarrow & H \rightarrow 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \\ 0 & \rightarrow & Y' & \longrightarrow & B' & \rightarrow & H' \rightarrow 0 \end{array}$$

commutes, where $B, B' \in dg\tilde{B}_N$ and $H, H' \in \tilde{L}_N$ are such that the rows are exact.

IV. THE BALANCE RESULT

Now let (A, L) and (L, B) are two complete hereditary cotorsion pairs in Φ . It would be interesting to know if one has balancedness in relative cohomology. The aim of this section is to show that the relative cohomology functor we discussed is balanced.

The following definition from [20, Defintion 3.1]

Definition 4.1 Let X and Y be N -complexes. A morphism $f: X \rightarrow Y$ induces homomorphisms $H_n^t(f): H_n^t(X) \rightarrow H_n^t(Y)$ for all $n \in Z$ and $t = 1, \dots, N-1$. We say that f is a quasi-isomorphism if each $H_n^t(f)$ is an isomorphism.

Lemma 4.2 Let X and Y be N -complexes. Then

$$H_n^t Hom_{\Phi}(X, Y) \cong Hom_{K_N(\Phi)}(X, \theta^{-n}Y).$$

In particular, $Hom_{\Phi}(X, Y)$ is exact if and only if $Hom_{K_N(\Phi)}(X, Y)$, i.e. $H_n^t Hom_{\Phi}(X, Y) = 0$ is equivalent to $Hom_{K_N(\Phi)}(X, \theta^{-n}Y) = 0$ for $t = 1, \dots, N-1$.

Remark The above lemma 4.2 from [13, lemma 4.4], where the $K_N(\Phi)$ denote the homotopy category of N -complexes, and the θ is a shift functor (see [9]).

Theorem 4.3 Let (A, L) and (L, B) be complete hereditary cotorsion pairs in Φ . Then, for any N -complex X and Y and for all $n \in Z, t = 1, \dots, N-1$. there exists an isomorphism

$$Ext_{dg\tilde{A}_N}^n(X, Y) \cong Ext_{dg\tilde{B}_N}^n(X, Y).$$

Proof Since (X, L) is a complete hereditary cotorsion

pair in Φ , we know $(dg\tilde{A}_N, \tilde{L}_N)$ is a complete hereditary cotorsion pair in $C_N(\Phi)$. Thus there exists an exact sequence $0 \rightarrow L \rightarrow A \rightarrow X \rightarrow 0$ with $A \in dg\tilde{A}_N$ and $L \in \tilde{L}_N$. Similarly since (L, B) is a complete hereditary cotorsion pair in Φ , we know $(\tilde{L}_N, dg\tilde{B}_N)$ is a complete hereditary cotorsion pair in $C_N(\Phi)$. Thus there exists an exact sequence

$$0 \rightarrow Y \rightarrow B \xrightarrow{\tau} H \rightarrow 0$$

With $B \in dg\tilde{B}_N$ and $H \in \tilde{L}_N$. Next we show that $H_n^1 Hom_\Phi(A, Y) \cong H_n^1 Hom_\Phi(X, B)$, it is just to prove that $H_n^1 Hom_\Phi(A, Y) \cong H_n^1 Hom_\Phi(X, B)$ by lemma 4.2. Note that for each $m \in Z$, we have

$$Hom_\Phi(A, \tau)_m = \prod_{i \in Z} Hom_\Phi(A_i, \tau_{i+m}).$$

We claim that

$$Hom_\Phi(A_i, \tau_{i+m}) : Hom_\Phi(A_i, B_{i+m}) \rightarrow Hom_\Phi(A_i, H_{i+m})$$

is an epimorphism for any $m \in Z$. In fact, suppose that $f \in Hom_\Phi(A_i, H_{i+m})$. Consider the exact sequence $0 \rightarrow K \rightarrow P \rightarrow H_{i+m} \rightarrow 0$, since (L, B) is a complete hereditary cotorsion pair, it follows that $K \in L$, which implies $Ext_\Phi^1(A_i, K) = 0$. Thus there exists $g : A_i \rightarrow P$ such that the following diagram commutes

$$\begin{array}{ccccccc} & & A_i & \rightarrow & A_i & & \\ & & \downarrow g & & \downarrow f & & \\ 0 & \rightarrow & K & \rightarrow & P & \rightarrow & H_{i+m} \rightarrow 0 \\ & & & & \downarrow h & & \downarrow = \\ 0 & \rightarrow & Y_{i+m} & \rightarrow & B_{i+m} & \rightarrow & H_{i+m} \rightarrow 0 \end{array}$$

This means that

$$Hom_\Phi(A_i, \tau_{i+m}) : Hom_\Phi(A_i, B_{i+m}) \rightarrow Hom_\Phi(A_i, H_{i+m})$$

is an epimorphism for any $m \in Z$. Hence, the homomorphism of complexes

$$Hom_\Phi(A, \tau) : \prod_{i \in Z} Hom_\Phi(A_i, B_{i+m}) \rightarrow \prod_{i \in Z} Hom_\Phi(A_i, H_{i+m})$$

is an epimorphism for any $m \in Z$. Thus the sequence of N -complexes

$$0 \rightarrow Hom(A, Y) \rightarrow Hom(A, B) \rightarrow Hom(A, H) \rightarrow 0$$

is exact.

Note that $A \in dg\tilde{A}_N$ and $H \in \tilde{L}_N$. Thus $Hom_\Phi(A, H)$ is an exact N -complex. Hence the morphism $Hom_\Phi(A, Y) \rightarrow Hom_\Phi(A, B)$ is a quasi-isomorphism. Similar discussion will yield that the morphism $Hom_\Phi(X, B) \rightarrow Hom_\Phi(A, B)$ is quasi-isomorphism. Thus the desired conclusion $H_n^1 Hom_\Phi(A, Y) \cong H_n^1 Hom_\Phi(X, B)$ follows.

Corollary 4.4 Let (A, L) and (L, B) be complete hereditary cotorsion pairs in Φ and X and Y be N -complexes. If A is a special $dg\tilde{A}_N$ -precover of X and B is a special $dg\tilde{B}_N$ -preenvelope of Y then, for any $n \in Z$ and $t = 1, \dots, N-1$, there exists an isomorphism $Ext_{dg\tilde{A}_N}^n(X, Y) \cong H_n^t Hom_\Phi(A, B)$.

Let (A, L) and (L, B) be complete hereditary cotorsion pairs in Φ . Then (A, L, B) is called a cotorsion triple. This notion was introduced recently by Chen [4]. The cotorsion triple (A, L, B) is said to be hereditary (resp. complete) provided that both of the two cotorsion pairs (A, L) and (L, B) are hereditary (resp. complete). For any N -complexes X and Y and for any $n \in Z$, we define the n th relative cohomology group based on a complete hereditary cotorsion triple (A, L, B) as

$$Ext_{(A,L,B)}^n(X, Y) = Ext_{dg\tilde{A}_N}^n(X, Y) = Ext_{dg\tilde{B}_N}^n(X, Y).$$

By lemma 4.3, the n th relative cohomology group $Ext_{(A,L,B)}^n(X, Y)$ can be computed by a special $dg\tilde{A}_N$ -precover of the first variable $dg\tilde{B}_N$ -preenvelope of the second variable.

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