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Quantum State Estimation for Abstract Quantum Walk in Terms of Quantum Bernoulli Noise

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Abstract—**Quantum state estimation is the process of parameter estimation using relevant properties of quantum systems and quantum mechanics. It focuses on making the estimation schemes. In quantum systems, quantum states contain important information about the system and are related to unknown parameters. In this paper, we propose an estimation scheme for Abstract quantum walk in terms of quantum Bernoulli noise and assess the merits of this scheme using quantum state as a probe.**

***Keywords*—quantum Bernoulli noise; Abstract quantum walk; quantum estimation**

I. INTRODUCTION

Quantum statistics [1, 2] is an emerging branch of quantum technology that combines statistical inference with quantum mechanics to study precision measurements. The main goal of quantum state estimation [3] is to improve the accuracy [4] of the estimation and to use quantum methods in this process to achieve the goal. The quantum state estimation process is generally divided into initial state preparation, parameterization, measurement and data analysis, where parameterization refers to the dynamic evolution of the parameter to be estimated in dependence. Positive operator-valued measures act as measurement tool. A measurement changes the state of the system drastically. Therefore an identically prepared other system is used for the next measurement.

As quantum analogs of classical random walk, quantum walk [6, 7] are widely used in many fields such as quantum information theory and quantum computing. It is used to describe the evolution behavior of quantum walkers, which is described by quantum states. This paper discusses the estimation of the Abstract quantum walk [8, 9, 10] in terms of the quantum Bernoulli noise. Using the quantum states as probe [11], a specific estimation scheme is given and shown to be unbiased and consistent.

II. GENERAL THEORY OF QUANTUM STATE ESTIMATION

The purpose of quantum state estimation is to determine the state of a quantum system by means of certain measurements. Quantum measurement is an important way to obtain information about the parameters of a quantum system. Unlike classical measurements, quantum measurements cause a change in the quantum state, resulting in a completely different quantum state from the initial state, which is irreversible and requires a constant homogeneous preparation of the original quantum system for the next measurement.

Let $\mathcal{H}$ be a complex Hilbert space to describe a quantum system on which the density operator can represent the states of the quantum system. Let $d \geq 2$ be a positive integer, then the tensor product space $\mathcal{H}^{\otimes d}$ can be considered as the " d − reconstant identically prepared other system " of the system $\mathcal{H}$. If ρ is a state on $\mathcal{H}$, then $\rho^{\otimes d}$ as a state on $\mathcal{H}^{\otimes d}$ can be considered as " d −reconstant identically preparation" of ρ. Denote by $\mathfrak{B}(\mathcal{H}^{\otimes d})$ the set of all bounded operators on $\mathcal{H}^{\otimes d}$.

Definition 2.1 Let $d \geq 1$ and $\mathcal{X}_d$ be a finite or countably- infinite set associated with d. A positive operator-valued measure defined on $\mathcal{X}_d$ and valued in $\mathfrak{B}(\mathcal{H}^{\otimes d})$ is a mapping $\mathcal{M}_d(x)\colon \mathcal{X}_d \to \mathfrak{B}(\mathcal{H}^{\otimes d})$ such that $\mathcal{M}_d(x) \geq 0$, $\forall\, x \in \mathcal{X}_d$ and

$$\sum_{x\in\mathcal{X}_d} \mathcal{M}_d(x) = I, \qquad (2.1)$$

where I denotes the identity operator on $\mathcal{H}^{\otimes d}$.

Measurements are generally described by positive operator-valued measures in quantum measurements. Projection operator-valued measures are a special class of positive operator-valued measures, which describe measurements as von Neumann measurement. The quantum state of a quantum system is often related to some unknown parameter. Suppose that the quantum state is $\{\rho_\theta | \theta \in \Theta\}$. Further, only when the values of the parameters are known can detailed information about the system be obtained, so determining the values of the parameters is the basis for studying the system. The aim of state estimation is to obtain the value of the parameter as accurately as possible from the measurements performed on the system. To estimate the unknown parameter θ, it is necessary to give the corresponding estimation scheme.

Given a state ρ_θ on $\mathcal{H}$ and a positive operator-valued measure $\mathcal{M}_d\colon \mathcal{X}_d \to \mathfrak{B}(\mathcal{H}^{\otimes d})$, the result of measurement is a $\mathcal{X}_d$ −valued random variable by Bohr’s rule. If $x \in \mathcal{X}_d$, then the probability of obtaining a measurement of x with a parameter θ to be

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measured is

$$p_{d,\rho_\theta}(x) = \mathrm{Tr}[\rho_\theta \mathcal{M}_d(x)],$$

(2.2)

where $\mathrm{Tr}[\cdot]$ denotes the trace operation on the operator. By the operation property of the trace class operator, ρ_θ has the following trace relations with its "d −reconstant identically prepared other system "

$$\mathrm{Tr}(\rho_\theta^{\otimes d}) = \mathrm{Tr}(\rho_\theta)^d.$$

(2.3)

A mapping $\hat{\theta}_d : \mathcal{X}_d \to \Theta$ is an estimator of the unknown parameter $\theta \in \Theta$. Given an estimator $\hat{\theta}_d$, if the measurement in the state ρ_θ is $x \in \mathcal{X}_d$, then it follows that $\theta = \hat{\theta}_d(x)$. Before the measurement the state of the quantum system can be deduced to be $\rho_{\hat{\theta}_d(x)}$.

Definition 2.2 Let the sequence $\{\mathcal{M}_d, \hat{\theta}_d\}$ be the estimation scheme for the above unknown parameter $\theta \in \Theta$, where $d \geq 1$ and $\mathcal{M}_d : \mathcal{X}_d \to \mathfrak{B}(\hbar^{\otimes d})$ is the positive operator-valued measure and $\hat{\theta}_d : \mathcal{X}_d \to \Theta$ is an estimator.

If the estimation scheme $\{\mathcal{M}_d, \hat{\theta}_d\}$ satisfies

$$\sum_{x \in \mathcal{X}_d} \hat{\theta}(x) \mathrm{Tr}[\rho_\theta \mathcal{M}_d(x)] = \theta,$$

(2.4)

then the estimation scheme is said to be unbiased. The estimation scheme $\{\mathcal{M}_d, \hat{\theta}_d\}$ is said to be consistent if for each $\theta \in \Theta$, there is

$$\lim_{n\to\infty} \sum_{x \in \mathcal{X}_d} e^{it\hat{\theta}}(x)\, p_{d,\rho_\theta}(x) = e^{it\theta}, t \in \mathbb{R},$$

(2.5)

where i is the imaginary unit.

III. Estimation of Abstrsct Quantum Walk in Terms of Quantum Bernoulli Noise

Quantum Bernoulli noises (QBN) are the family of annihilation and creation operators acting on the space $\hbar$ of square integrable Bernoulli functionals. They satisfy a canonical anti-commutation relation (CAR) in equal time and can play an important role in describing the environment of an open quantum system. Putting $\Xi_k = \partial_k + \partial_k^*$, the sum of the creation and annihilation operators in QBN, one gets a sequence $\{\Xi_k | k \geq 0\}$of unitary involutions (self-adjoint unitary operators) on $\hbar$, which we call the canonical unitary involutions associated with QBN. The following perturbations of the canonical unitary involutions

$$\mathrm{U}_k = \Xi_k(2|Z_k><Z_k| - I), \mathrm{k} \geq 0 \qquad (3.1)$$

can be used as an evolution operator for a class of Abstract quantum walk, where Z_k is the basis vector of the canonical orthonormal basis for $\hbar$, $|Z_k><Z_k|$ is the Dirac operator. By the spectral mapping theorem for Abstract quantum walk (see [10]) we can obtain

$$\sigma(U_k) = \varphi^{-1}(\{0\}) \cup \{1\}^{+\infty} \cup \{-1\}^{+\infty}$$
$$= \{i, -i\} \cup \{1\}^{+\infty} \cup \{-1\}^{+\infty},$$

where $\sigma(U_k)$ is used to mean the spectrum of U_k.

The quantum walk driven by U_k take the space $\hbar$ as the state space, there exists unit vector $\xi_t \in \hbar$, satisfying the following evolution equation

$$\xi_t = {U_k}^t \xi_0, \ \ t \geq 0,$$

(3.2)

while ξ_t is called its state at time t, in particular ξ_0 is known as the initial state.

For $t \geq 0$,

$$\sum_{\tau\in\Gamma} |< Z_n, \xi_t >|^2 = \| \xi_t \|^2 = \| {U_k}^t \xi_0 \|^2 = \| \xi_0 \|^2 = 1, \qquad (3.4)$$

the function $\tau \to |< Z_n, \xi_t >|^2$ defines a probability distribution on Γ, which is known as the probability distribution of the Abstract quantum walk at time $t \geq 0$.

Lemma 3.1 [10] U_k has an invariant probability distribution of the following form

$$\mu_k(\tau) = \begin{cases} \frac{1}{2}, \tau \in \{k\}; \\ \frac{1}{2}, \tau \in \phi\,; \\ 0, \tau \in \Gamma \backslash \{\phi, \{k\}\}. \end{cases} \qquad (3.5)$$

Proof: Let $\xi = \frac{\sqrt{2}}{2}(Z_k + iZ_\emptyset)$, then ξ is the unit vector in $\hbar$. We know that $U_k \xi = -i\xi$. For all $t \geq 0$, we can obtain by calculation

$$|< Z_\tau, U_k^t \xi >|^2 = |< Z_\tau, (-i)^t \xi >|^2$$
$$= \frac{1}{2} |< Z_\tau, Z_k + iZ_\emptyset >|^2$$
$$= \begin{cases} \frac{1}{2}, \tau \in \{k\}; \\ \frac{1}{2}, \tau \in \phi\,; \\ 0, \tau \in \Gamma \backslash \{\phi, \{k\}\}. \end{cases}$$

It follows that for all $\tau \in \Gamma$ and $t \geq 0$, we have $\mu_k(\tau) = |< Z_\tau, U_k^t \xi >|^2$, that is, the function μ_k defined by (3.5) is an invariant probability distribution of U_k.

As seen above, the invariant probability distribution of Abstract quantum walk driven by $\mathrm{U}_k = \Xi_k(2|Z_k><Z_k| - I)$ associate with k. Consider the unitary operator U_k with the unknown parameter $k \geq 0$ and use the general principle of quantum state estimation to establish an estimation scheme for the unknown parameter $k \geq 0$ in the above Abstract quantum walk.

Proposition 3.1 Let $\mathfrak{B}(\hbar)$ be the set of all bounded operators on $\hbar$ and $\mathbb{N}$ a non-negative integer, define a mapping $\pi : \mathbb{N} \to \mathfrak{B}(\hbar)$ as

$$\pi(\mathrm{n}) = |Z_n >< Z_n|,\ \ \mathrm{n} \in \mathbb{N},\qquad \#\tau = \mathrm{n},\ \tau \in \Gamma. \qquad (3.6)$$

Then π is a projection operator-valued measure.

Proof: For each $\mathrm{n} \in \mathbb{N}$, since Z_n is the basis vector of the standard orthogonal basis vector for $\hbar$,$\pi(\mathrm{n})$ is the projection operator on $\hbar$. Let $\xi \in \hbar$, then

$$\sum_{n\in N} \pi(\mathrm{n})\xi = \sum_{n\in N} |Z_n >< Z_n|\xi \ = \xi,$$

which shows that it holds in the strongly convergent sense

$$\sum_{n\in N} \pi(\mathrm{n}) = I,$$

where I is the identity operator on $\hbar$. Hence, π is a projection operator-valued measure.

Proposition 3.2 The state ξ_t of Abstract quantum walk at time t satisfies the following relationship with the projection operator value measure π,

$$\mathrm{Tr}[|\xi_t >< \ \xi_t|\pi(\mathrm{n})] = < \xi_t, \pi(\mathrm{n})\xi_t >_\hbar = ||\xi_t||^2,\ \ \mathrm{n} \in \mathbb{N}. \qquad (3.7)$$

Proof: The first equation holds from the operation properties of trace class operators. From the definition of $\pi(\mathrm{n})$ and $\xi_t \in \hbar$, we have

$$< \xi_t, \pi(\mathrm{n})\xi_t >_\hbar = < \xi_t, |Z_n >< Z_n|\xi_t > = |< Z_n, \xi_t >|^2 = ||\xi_t||^2.$$

Theorem 3.1 Let $\mathrm{d} \geq 1$ and $\mathfrak{B}(\hbar^{\otimes d})$ be the set of all bounded operators on $\hbar^{\otimes d}$. Suppose that

$$\pi^{\otimes d}(n) = \pi(\mathrm{n}_1)\otimes\pi(\mathrm{n}_2)\otimes\cdots\otimes\pi(\mathrm{n}_d), \mathrm{n} = (\mathrm{n}_1, \mathrm{n}_2, \cdots, \mathrm{n}_d) \in \mathrm{N}^d. \qquad (3.8)$$

Then the mapping $\pi^{\otimes d}: \mathrm{N}^{\otimes d} \to \mathfrak{B}(\hbar^{\otimes d})$ is a projection operator-valued measure.

Proof: For each $\mathrm{n} = (\mathrm{n}_1, \mathrm{n}_2, \cdots, \mathrm{n}_d) \in \mathbb{N}^d$, it follows from Proposition 3.2 and the tensor product property of the projection operator that $\pi^{\otimes d}(n)$ is a projection operator in the tensor product space $\mathfrak{B}(\hbar^{\otimes d})$. Let $n^{(1)}, n^{(2)} \in \mathbb{N}^d$ and $n^{(1)} \neq n^{(2)}$. Then we have the indicator $\mathrm{j} \in \{1,2,\cdots,\mathrm{d}\}$ such that $n_j^{(1)} \neq n_j^{(2)}$. Thus $\pi(n_j^{(1)})\pi(n_j^{(2)}) = 0$. It shows

$$\begin{aligned}&\pi^{\otimes d}(n^{(1)})\pi^{\otimes d}(n^{(2)})\\ &= \pi(\mathrm{n_1}^{(1)})\pi(\mathrm{n_1}^{(2)})\otimes\cdots\otimes\pi(\mathrm{n}_j{}^{(1)})\pi(\mathrm{n}_j{}^{(2)})\otimes\cdots\\ &\qquad\otimes\pi(\mathrm{n}_d{}^{(1)})\pi(\mathrm{n}_d{}^{(2)})) = 0.\end{aligned}$$

Let $\xi_j \in \hbar$, $1 \leq \mathrm{j} \leq \mathrm{d}$. We can calculate

$$\begin{aligned}\sum_{\mathrm{n}\in N^d} \pi^{\otimes d}(n)\left(\otimes_{j=1}^d \xi_j\right) &= \sum_{\mathrm{n}\in N^d} \otimes_{j=1}^d [\pi(n_j)\xi_j]\\ &= \otimes_{j=1}^d \sum_{n_j\in N^d} \pi(n_j)\xi_j = \otimes_{j=1}^d \xi_j,\end{aligned}$$

where $\mathrm{n} = (\mathrm{n}_1, \mathrm{n}_2, \cdots, \mathrm{n}_d)$. Since $\{\otimes_{j=1}^d \xi_j | \xi_j \in \hbar, 1 \leq \mathrm{j} \leq \mathrm{d}\}$ is a complete set of $\hbar^{\otimes d}$ and the sum sequences of the projection operator series part are uniformly bounded. Thus it follows that $\sum_{\mathrm{n}\in N^d} \pi^{\otimes d}(n) = I$, where I is the identity operator on $\hbar^{\otimes d}$. This completes the proof.

The invariant probability distribution of Abstract quantum walk in terms of quantum Bernoulli noise at time $\mathrm{t} \geq 0$ is $\mu_k(\tau): \tau \to | < Z_\tau, \xi_t > |^2$. For each $\mathrm{n} \in \mathbb{N}$,

its corresponding mean value

$$\hbar = \sum_{\#\tau=\mathrm{n},\tau\in\Gamma} n| < Z_\tau, \xi_t > |^2\ . \qquad (3.9)$$

This shows that the mean of the probability distribution of the Abstract quantum walk to be estimated at time t is a function of the unknown parameter $k \geq 0$. Thus, the estimation of the unknown parameter k can be transformed into estimation of its function $\hbar$.

To estimate the function of the unknown parameter k is estimation scheme. Consider the state $\rho_\hbar = |\xi_t >< \ \xi_t|$ at time t. For $\mathrm{d} \geq 1$, the quantum measurement described by the projection operator-valued measure $\pi^{\otimes d}$ is implemented under the state $\rho_\hbar^{\otimes d}$ and the result is a $\mathbb{N}^d$-valued random variable. Corresponding to the above measure, an estimator $\hat{\hbar}_d$ is chosen for $\hbar$ as follows

$$\hat{\hbar}_d = \frac{1}{d}\sum_{i=1}^d n_i, \qquad (3.10)$$

where $\mathrm{n} = (\mathrm{n}_1, \mathrm{n}_2, \cdots, \mathrm{n}_d) \in \mathbb{N}^d$ is the measurement result.

Theorem 3.2 The estimation scheme $\{\pi^{\otimes d}, \hat{k}_d\}$ is unbiased and consistent.

Proof: According to Theorem 3.1 and the relation $\sum_{\tau\in\Gamma} |<Z_\tau, \xi_t>|^2 = 1$, we obtain

$$\sum_{n\in N^d} \hat{k}_d(n) Tr\left[\rho_\gamma^{\otimes d}\pi^{\otimes d}(n)\right]$$

$$= \sum_{n\in N^d} (\frac{1}{d}\sum_{i=1}^{d} n_i) Tr\left[\otimes_{j=1}^{d}(\rho_k \pi(n_j))\right]$$

$$= \sum_{n\in N^d} (\frac{1}{d}\sum_{i=1}^{d} n_i \prod_{j=1}^{d} Tr\left[\rho_k \pi(n_j)\right])$$

$$= \sum_{n\in N^d} (\frac{1}{d}\sum_{i=1}^{d} n_i \prod_{j=1}^{d} ||\xi_t(n_j)||^2)$$

$$= \frac{1}{d}\sum_{n\in N^d}\sum_{i=1}^{d} n_i \prod_{j=1}^{d} \sum_{\#\tau=n,\tau\in\Gamma} \mu_k(\tau)$$

$$= \frac{1}{d}\sum_{i=1}^{d} k = k.$$

Therefore the above estimation scheme is unbiased. On the other hand, for the d-dimensional random vector $(\eta_1, \eta_2, \cdots, \eta_d)$ we have

$$F(\eta_1 = n_1, \eta_2 = n_2, \cdots, \eta_d = n_d)$$

$$= \prod_{j=1}^{d} ||\xi_t(n_j)||^2, (n_1, n_2, \cdots, n_d) \in \mathbb{N}^d.$$

The random variables $\eta_1, \eta_2, \cdots, \eta_d$ are independent of each other and follow the same distribution. The expectation and variance of η_1 can be calculated as $E(\eta_1) = k$ and

$$\mathrm{Var}(\eta_1) = E\eta_1^2 - (E\eta_1)^2 = 1 - k^2.$$

By the Large Number Theorem, $\frac{1}{d}\sum_{i=1}^{d} n_i$ converges to k with probability, which further gives

$$\sum_{N\in N^d} e^{it\hat{k}_d(N)} Tr\left[\rho_\gamma^{\otimes d}\pi^{\otimes d}(N)\right]$$

$$= \sum_{N\in N^d} e^{\frac{it}{d}\sum_{j=1}^{d} n_j} \prod_{j=1}^{d} ||\xi_t(n_j)||^2$$

$$= E e^{\frac{it}{d}\sum_{j=1}^{d} n_j} \to e^{itk}, \forall t \in \mathbb{R}.$$

Hence, the above estimation scheme is consistent.

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