The CEV-SV Model–Portfolio Optimization With CARA Utility Under Two Different Cases Of Prices Of Risk

Yang Xiaoye and Xiao Hongmin

Abstract— **In this paper,we consider a new diffusion models for stock prices with applications in portfolio optimization to overcome the main restriction of the CEV model. The diffusion model combines constant elasticity of volatility(CEV) and stochastic volatility (SV) to create the CEV-SV model,while the SV component features the state-of-the-art 4/2 model.So that the complete correlation between the price of risky asset and its volatility is decoupled, and added randomness between the price of risky assets and their volatility.It can be clearly noticed that when $\beta = 0$,the CEV-SV Model degenerates to SV Model.We study an investment problem within expected utility theory (EUT) for incomplete markets, producing closed-form representations for the optimal strategy for two different cases of prices of risk on the stock.Finally,numerical examples are provided to support our theoretical results.We find that under the first risk price, the optimal investment strategy exhibits reduced stability and is more susceptible to stock price volatility.Therefore, it is advisable for different investors to adopt distinct investment strategies. Specifically, risk-averse or neutral investors may find it suitable to invest under the second risk price, while risk-loving investors may prefer investing under the first risk price.And the numerical simulation also tells us that the presence of CEV component incites a sharp downward movements in the optimal allocation toward short maturities.**

Index Terms—**CEV-SV model, expected utility theory, two different cases of prices of risk.**

I. Introduction

The literature on optimal investment within expected utility began in the late 60s,where the risky asset price was assumed as a geometric Brownian motion (GBM); see the celebrated work of Merton (1969)[1].Since then,many empirical studies have shown that this simple model cannot properly fit real market data. The main drawback is that the GBM does not capture implied volatility smile/skew effects from option prices.To address its limitation,a simple extension of the GBM is the so-called local-volatility constant elasticity of variance (CEV) model,originally proposed by Cox (1975)[2] and Cox and Ross (1976)[3] as an alternative diffusion process for European option pricing.Compared with the GBM,the merits of the CEV model are that the volatility rate correlates with the risky asset price,known as the leverage effect,and empirical biases such as volatility smile can be better captured.Beckers(1980)[4],MacBeth and Merville (1980)[5],and Emanuel and MacBeth(1982)[6] presented some theoretical arguments as well as empirical evidence to support volatility changing with the stock price and a negative elasticity factor.The main application of the CEV process has been on derivative pricing (see,for instance,Cox (1996)[7]);Davydov and Linetsky 2001[8];Lo et al.2000[9]).As for portfolio optimization,Gao(2009)(10) explored the optimal investment problem for defined contributions (DC) retirement plans under a CEV model.The author derived the respective explicit solutions for the CRRA and CARA utility functions by applying the stochastic optimal control,power transformation,and variable change technique. However,one main disadvantage in the CEV framework is that volatilities and underlying risky asset prices are perfectly correlated.Much evidence currently exists of volatilities being correlated with but decoupled from stock prices (e.g., implied volatility structures and trading of volatility indexes).These stylized facts cannot be captured within constant elasticity of volatility models.To overcome this restriction,some researchers have proposed a time-varying elasticity parameter β (see Ghysels et al. 1996[11]; Harvey 2001[12]). In particular, Kim et al. (2014)[13] introduced an extension of the CEV model named “stochastic elasticity of variance”(SEV). The authors relaxed the time deterministic elasticity assumption by allowing the elasticity to vary randomly,and hence decoupled the movements of implied volatility from the risky asset prices.

In our research,inspired by Escobar-Anel et al.(2023)[14],we have developed a new approach to overcome the main restriction of the CEV model by combine constant elasticity of volatility (CEV) and stochastic volatility (SV) to create the CEV-SV model,while the SV component features the state-of-theart 4/2 model.So that it can better reflect empirical deviations such as the volatility smile in the market by linking volatility with risky asset prices.At the same time,it uses different characteristic factors to describe the implied volatility, which overcomes the shortcoming of the CEV model that volatility is completely related to risky asset prices.As is well known,SV models are capable of generating random volatility correlated with the stock price. The 4/2 model is a new popular stochastic volatility (SV) model that was first introduced by (Grasselli 2017[15]). It combines the classic (Heston 1993[16]) i.e.,the 1/2 (Heston) model and the 3/2 model of (Heston 1997[17] and (Platen 1997[18]).By combining the two,the 4/2 model inherits the benefits of both 1/2 and 3/2 models while bringing additional benefits

Yang Xiaoye, College of Mathematics and Statistics,Northwest Normal University,Lanzhou Gansu,730070, China.

Xiao, Hongmin, Professor of Northwest Normal University,Lanzhou G ansu,730070, China.

such as an instantaneous volatility uniformly bounded away from zero and closed-form solutions for pricing derivatives (see, e.g.,Cui et al.(2017)[19], (2018)[20]).The diffusion part of the model is seen as the superposition of two independent volatility series. The first series is formed by the CIR process,and the second series follows a 3/2 diffusion.So that the implied volatility can be described by factors with different characteristics,with the former having a mean-reversion that is independent of the factor level,and the latter being empirically more reasonable. Our new hybrid model is forming by introducing an instantaneous volatility of the form $S_t^{\beta}(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})$ for some constants a, b ,where v_t is the CIR factor.We denote this hybrid structure as the CEV-4/2 model.Besides,we introduced a new form of risk asset interest rates: $r+(\bar{\lambda}\sqrt{v_t}+\bar{\lambda}_c)S_t^{\beta}(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})$,where $\bar{\lambda}\geq 0$; $\bar{\lambda}_c\geq 0$. $\bar{\lambda}\sqrt{v_t}+\bar{\lambda}_c$ denotes the MPR (two choices of prices of risk).In this form, we link the interest rate of risky assets to the price and the volatility of risky assets,which is also in line with the real market.Generally speaking, high-price risky assets are always accompanied by high interest rates.The cases $\bar{\lambda}=0$ and $\bar{\lambda}_c=0$ will be considered separately.

1. $\bar{\lambda}_c=0$ and $\bar{\lambda}\neq 0$.In this setting,the interest rate of risky assets becomes

$$r+\bar{\lambda}S_t^{\beta}(av_t+b).$$

2. $\bar{\lambda}_c\neq 0$ and $\bar{\lambda}=0$.In this setting,the interest rate of risky assets becomes

$$r+\bar{\lambda}_c S_t^{\beta}(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}}).$$

In the first risk price, the risk asset interest rate is only controlled by the term v_t ;while in the second risk price, the risk asset interest rate is controlled by the 4/2 term. Therefore, under the second risk price, the risk asset interest rate is more stable. The main contributions of the paper are as follows:

1. we define a new diffusion models for stock prices with applications in portfolio optimization to overcome the main restriction of the CEV model ;
2. In closed-form optimal investment, we find optimal wealth and value function for a risk-averse investor within expected utility theory, for two choices of prices of risk (MPR).

The rest of this paper is organized as follows.In Section 2,we set up the financial market structure and the investment problem;then derive a HJB equation by applying the dynamic programming principle. In Section 3, we obtain the optimal strategy under two different risk prices within expected utility theory.Then numerical analysis were conducted on the optimal strategy.Section 4 is the conclusions.

II. Materials and Methods

A. Model Formulation:The optimization problem

Assume that a financial market consists of one risk-free asset and one risky asset (i.e.,stock),Let all the stochastic processes introduced in this paper be defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\in[0,T]})$,where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a right-continuous filtration generated by standard Brownian motions (BMs). We assume that the price process of the risk-free asset B_t evolves according to

$$dB_t=rB_t dt, \qquad (1)$$

where the interest rate r is assumed to be constant. The price processes S_t of the risky asset follows the hybrid structure of what we defined as an CEV-4/2 model

$$\frac{dS_t}{S_t}=[r+(\bar{\lambda}\sqrt{v_t}+\bar{\lambda}_c)S_t^{\beta}(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})]dt$$
$$+S_t^{\beta}(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})(\rho dZ_{1t}+\sqrt{1-\rho^2}dZ_{2t}), \qquad (2)$$

where $s_0>0$. The stochastic factor v_t satisfies the following stochastic differential equation(SDE)

$$dv_t=\kappa(\theta-v_t)dt+\sigma\sqrt{v_t}dZ_{1t}, \quad v_0>0, \qquad (3)$$

With $r,\theta,\kappa,\sigma\in\mathbb{R}_+$; $\beta\leq 0$; $\bar{\lambda}\geq 0$; $\bar{\lambda}_c\geq 0$; $a\geq 0$; and $b\geq 0$ .Moreover, s_0, v_0 are initial values; Z_1, Z_2 are independent Brownian motions. In this setting, the MPR becomes $(\bar{\lambda}\sqrt{v_t}+\bar{\lambda}_c)$; therefore the cases $\bar{\lambda}=0$ and $\bar{\lambda}_c=0$ will be considered separately.

Our model permits a risk-neutral pricing measure $\mathbb{Q}$, which we identify via the change of measure:

$$\begin{aligned} dZ_{1,t}&=dZ_{1,t}^{\mathbb{Q}}-\lambda_1\sqrt{v_t}dt\\ dZ_{2,t}&=dZ_{2,t}^{\mathbb{Q}}-(\lambda_2\sqrt{v_t}+\lambda_{2,c})dt \qquad (4)\\ dZ_{3,t}&=dZ_{3,t}^{\mathbb{Q}}-\lambda_{3,c}dt,\end{aligned}$$

Where $dZ_{i,t}$, $i=1,\ldots,3$ are independent standard Brownian motions under $\mathbb{Q}$.This means,as per the notation in (2), that $\bar{\lambda}=\rho\lambda_1+\sqrt{1-\rho^2}\lambda_2$ and $\bar{\lambda}_c=\sqrt{1-\rho^2}\lambda_{2,c}$,

Proposition 1. The following conditions are needed for the change of measure in (4) to be well defined:

$$\begin{gathered} 2\kappa\theta\geq\sigma^2\\ \kappa+\lambda_1\sigma>0 \qquad (5)\\ \max|\lambda_1|,|\lambda_2|<\frac{\kappa}{\sigma}\end{gathered}$$

See Escobar-Anel et al.2023 for a proof.

B. Hamilton-Jacobi- Bellman Equations

We invest in the stock and a cash account B_t;$\pi_t$is the proportion of wealth allocated to the stock, and $(1-\pi_t)$hence goes to cash. Using the self-financing condition, the wealth process for this investor under the real-world measure $\mathbb{P}$ is given by

$$\begin{aligned}\frac{dX_t}{X_t}&=\pi_t\frac{dS_t}{S_t}+(1-\pi_t)\frac{dB_t}{B_t}\\ &=[r+\pi_t(\bar{\lambda}\sqrt{v_t}+\bar{\lambda}_c)S_t^{\beta}(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})]dt \qquad (6)\\ &+\pi_t[S_t^{\beta}(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}})(\rho dZ_{1t}+\sqrt{1-\rho^2}dZ_{2t})],\end{aligned}$$

where $x_0=x>0$ is an initial wealth.

For all $(x_0,v_0)\epsilon\mathbb{R}^+\times\mathbb{R}^+$ and $t\epsilon[0,T]$,we assume that the (6) has a pathwise unique solution $\{X_t^{\pi}\}_{t\epsilon[0,T]}$ under the real-world measure $\mathbb{P}$. Let us define

$$U(x,v):=\{\pi:=(\pi_t)_{t\epsilon[0,T]}|\pi \text{ as progressive measures}, X(0)=x_0, v_0=v, \mathbb{E}^{\mathbb{P}}_{x_0,v_0,t_0}[u(X_t)]<\infty,\}, \qquad (7)$$

where $\mathbb{E}^{\mathbb{P}}_{x,v,t}[\cdot]=\mathbb{E}^{\mathbb{P}}[\cdot\,|X_t=x,v_t=v]$ denotes the conditional expectation.

To seek the optimal investment strategy π_t,we maximize the expected utility of the terminal wealth

$$V(t,x,v)=\max_{\pi_t\in U}\mathbb{E}_{t,x,v}[U(X_T)],\qquad 0<t<T \qquad (8)$$

where $U(\cdot)$ is the expected utility function.

$$U(x)=-\frac{e^{-qx}}{q},q\neq 0.$$

That is increasing and concave. $V(t,x,v)$ is the value function,and $U(x,v)$ denotes the space of admissible trading strategies.By using the dynamic programming approach,we obtain the form of optimal investment strategy π_t^* and the Hamilton-Jacobi-Bellman(HJB) equation for this optimization problem,with boundary condition

$$V(T,x,v)=-\frac{e^{-qx}}{q}.$$

To reduce the complexity of the problem,make a change of control:

$$\psi_t=\pi_t S_t^{\beta}(a\sqrt{v_t}+\frac{b}{\sqrt{v_t}}), \qquad (9)$$

The problem of interest under this new control becomes

$$V(t,x,v)=\max_{\pi}\mathbb{E}_{t,x,v}[U(X_T)]=\max_{\psi}\mathbb{E}_{t,x,v}[U(X_T)],$$

with wealth process as follows

$$\frac{dX_t}{X_t}=(r+\psi_t(\bar{\lambda}\sqrt{v_t}+\bar{\lambda}_c))dt+\psi_t(\rho dZ_{1t}+\sqrt{1-\rho^2}dZ_{2t})$$

$$dv_t=\kappa(\theta-v_t)dt+\sigma\sqrt{v_t}dZ_{1t},\qquad v_0>0. \qquad (10)$$

The HJB equation associated with the optimization problem is given by the partial differential equation(PDE)

$$\begin{aligned}0=\sup_{\psi}V_t+\kappa(\theta-v)V_v+\frac{1}{2}\sigma^2vV_{vv}+x(r+\psi(\bar{\lambda}\sqrt{v_t}\\+\bar{\lambda}_c))V_x+\frac{1}{2}x^2\psi^2V_{xx}+x\psi\sigma\rho\sqrt{v}V_{xv}\\=V_t+\kappa(\theta-v)V_v+\frac{1}{2}\sigma^2vV_{vv}\\+\sup_{\psi}x(r+\psi(\bar{\lambda}\sqrt{v_t}+\bar{\lambda}_c))V_x\\+\frac{1}{2}x^2\psi^2V_{xx}\\+x\psi\sigma\rho\sqrt{v}V_{xv}, \qquad (11)\end{aligned}$$

with the final condition $V(T,x,v)=U(x)$. Here,each V with the subscript denotes the partial derivative with respect to the corresponding variables.From (11),it is immediately observed that the first order maximizing condition for the optimal strategy ψ^* is given by

$$\begin{aligned}\psi^*=-\frac{x(\bar{\lambda}\sqrt{v_t}+\bar{\lambda}_c)V_x+x\sigma\rho\sqrt{v}V_{xv}}{x^2V_{xx}}\\=-\frac{(\bar{\lambda}\sqrt{v_t}+\bar{\lambda}_c)V_x}{\square V_{xx}}\\-\frac{\sigma\rho\sqrt{v}V_{xv}}{xV_{xx}}. \qquad (13)\end{aligned}$$

In the next section,we will obtain the optimal strategy under two different risk prices within expected utility theory,and illustrate the dynamics of optimal investment strategy .

III. Results

A. Portfolio Problem and Solution

To find the solution of (13),we use the separation ansatz

$$V(t,x,v)=-\frac{1}{q}\exp\{-q[a(t)(x-b(t))+h(t,v)]\}, \qquad (14)$$

with $a(T)=1,\quad b(T)=0,\ h(T,v)=0.$

We assume $b(t)=0$ in (14) for simplicity. Then,using

$$\begin{aligned}V_t=(a_tx+h_t)\exp\{-q(a(t)x+h(t,v))\}\\V_v=h_v\exp\{-q(a(t)+h(t,v))\}\\V_x=a(t)\exp\{-q(a(t)x+h(t,v))\} \qquad (15)\\V_{vv}=(h_{vv}-qh_v^2)\exp\{-q(a(t)x+h(t,v))\}\\V_{xv}=-qa(t)h_v\exp\{-q(a(t)x+h(t,v))\}\\V_{xx}=-qa(t)^2\exp\{-q(a(t)x+h(t,v))\}\end{aligned}$$

Thereby,substituting (15) into (12),we have

$$\psi^*=\frac{\bar{\lambda}\sqrt{v_t}+\bar{\lambda}_c}{a(t)\square q}-\frac{\sigma\rho\sqrt{v}h_v}{a(t)x} \qquad (16)$$

Substituting the ansatz into the HJB equation yields

$$\begin{aligned}a_tx+a(t)rx+h_t+\kappa(\theta-v)h_v+\frac{1}{2}\sigma^2vh_{vv}-\frac{1}{2}\sigma^2vq(1\\-\rho^2)h_v^2+\frac{1}{2q}(\bar{\lambda}\sqrt{v_t}+\bar{\lambda}_c)^2-\sigma\rho\sqrt{v}(\bar{\lambda}\sqrt{v_t}+\bar{\lambda}_c)h_v\\=0. \qquad (17)\end{aligned}$$

Further,we can decompose (17) into two equations

$$a_tx+a(t)rx=0, \qquad (18)$$

$$\begin{aligned}h_t+\kappa(\theta-v)h_v+\frac{1}{2}\sigma^2vh_{vv}-\frac{1}{2}\sigma^2vq(1-\rho^2)h_v^2\\+\frac{1}{2q}(\bar{\lambda}\sqrt{v_t}+\bar{\lambda}_c)^2-\sigma\rho\sqrt{v}(\bar{\lambda}\sqrt{v_t}\\+\bar{\lambda}_c)h_v\\=0, \qquad (19)\end{aligned}$$

with $a(T)=1,\ h(T,v)=0.$ The solution of (18) can be given by

$$a(t)=e^{r(T-t)} \qquad (20)$$

Proposition 2. The solution to (19) is provided next in two cases:

- Assume $\bar{\lambda}_c=0$ and $\bar{\lambda}\neq 0$;The value function (19) can be expressed as (21),

$$h(t,v)=K_1\kappa\theta(T-t), \qquad (21)$$

with

$$K_1=\frac{-(\kappa+\sigma\rho\bar{\lambda})\pm\sqrt{(\kappa+\sigma\rho\bar{\lambda})^2+\sigma^2q^2(1-\rho^2)\bar{\lambda}^2}}{\sigma^2q(1-\rho^2)}.$$

Then,

$$\psi_1^*=\frac{\bar{\lambda}\sqrt{v}}{e^{r(T-t)}xq}. \qquad (22)$$

Furthermore,the optimal strategy π_1^* under the real-world measure $\mathbb{P}$ is given by

$$\pi_1^*=\frac{\psi_1^*}{S^{\beta}(a\sqrt{v}+\frac{b}{\sqrt{v}})}=\frac{\bar{\lambda}\sqrt{v}}{S^{\beta}(a\sqrt{v}+\frac{b}{\sqrt{v}})e^{r(T-t)}xq}. \qquad (23)$$

- Assume $\bar{\lambda}_c\neq 0$ and $\bar{\lambda}=0$;The value function (19) can be expressed as (24),

$$h(t,v)=\frac{1}{2q}\bar{\lambda}_c^2(T-t) \qquad (24)$$

$$\psi_2^*=\frac{\bar{\lambda}_c}{e^{r(T-t)}xq} \qquad (25)$$

Furthermore,the optimal strategy π_2^* under the real-world measure $\mathbb{P}$ is given by

$$\pi_2^*=\frac{\psi_2^*}{S^{\beta}(a\sqrt{v}+\frac{b}{\sqrt{v}})}=\frac{\bar{\lambda}_c}{S^{\beta}(a\sqrt{v}+\frac{b}{\sqrt{v}})e^{r(T-t)}xq} \qquad (26)$$

Proof 1. The proof is presented in Appendix A.

B. Numerical analysis

From C,it can be seen that the optimal solution to the portfolio problem depends on the parameters of the model.We illustrate the dynamics of an optimal investment strategy π_1^* with CARA utility at time points $t=0,1,2$.In these examples,for convenience,we assume that all of the duration dates are T = 3 and $\bar{\lambda}_c = 0$. In our model, 9 parameters are required: $r, \bar{\lambda}, a, b, \rho, \beta, \kappa, \theta, \sigma$. We propose values for these parameters via a combination of two reliable sources on the embedded key models: the CEV and the 4/2.The hybrid structure not only possesses new features but also inherits some characteristics of each of the two models.The parameters $\bar{\lambda}, a, b, \rho, \kappa, \theta, \sigma$ are inherited from the 4/2 model,and their estimates were already obtained by (Cheng and Escobar-Anel 2021),and the parameters β was obtained by(Sung-Jin Yang and Kim 2013).

Parameters	Estimation
r	0.05
q	0.05
$\hat{\lambda}$	2.9428
a	0.9051
b	0.0023
ρ	0.7689
κ	7.3479
σ	0.6612
β	0.5
Theoretical leverage($v_t = \theta$)	-0.76889

Table1:Estimates among the various models.

Table. 1 displays these parameter values. Fig. 1-3 ,we present numerical analysis for the optimal investment strategy π_1^* by shifting the value of a certain parameter and holding all other parameters constant.

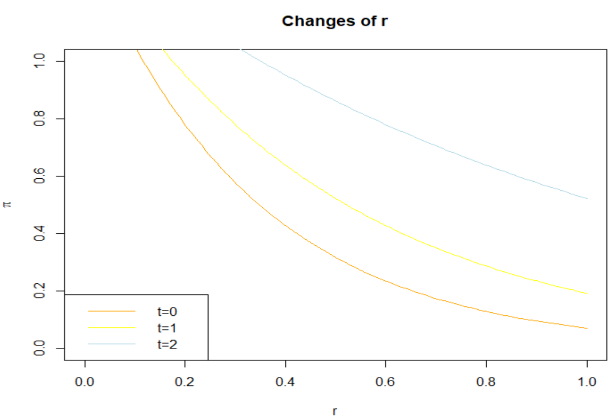

Figure 1: The impact of r on π_1^*

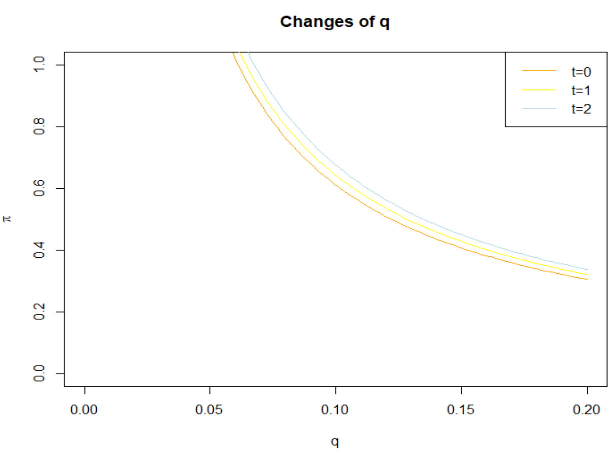

Figure 2: The impact of q on π_1^*

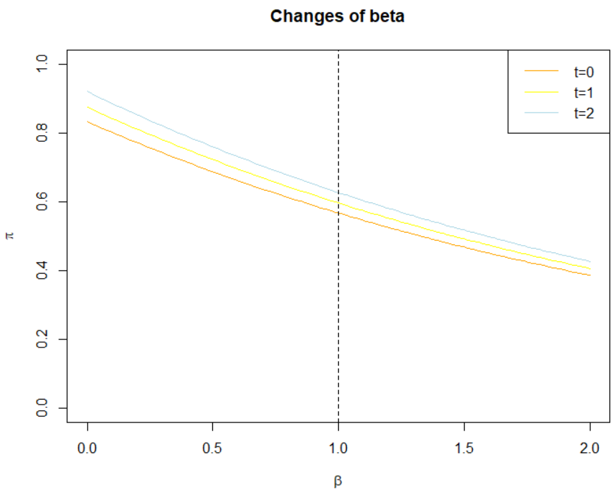

Figure 3: The impact of β on π_1^*

Fig. 1 plots the optimal proportion π_1^* by shifting parameter r, which shows the effect of interest rates on the value of π_1^* .Fig. 1 shows that as the interest rate increases,the optimal proportion of investment on the risky asset would decrease correspondingly.

In Fig. 2-3,parameter q denotes the risk aversion coefficient of investors,parameter β denotes the elasticity of volatility coefficient.These figures show that the larger the value of q and β,the smaller proportion of investment in the risky market.The investment proportion is a decreasing function of the risk aversion coefficient qand β.And the CEV-SV Model degenerate to SV Model when $\beta = 0$,which means the presence of CEV component incites a sharp downward movements in the optimal allocation toward short maturities.

IV. Summary and Conclusions

This paper presents the first portfolio optimization analysis using a hybrid structure of the CEV and SV models.We obtained closed-form solutions for the optimal strategy within EUT for two choices of prices of risk.And provide numerical analysis to illustrate the effect of the model parameters on the optimal investment strategy.The results shows that the optimal investment strategies calculated under two different risk prices were very similar, except that the optimal investment strategy under the first risk price had an additional factor of $\sqrt{v}$. This also suggests that under the first risk price, our optimal investment strategy is less stable and susceptible to stock price volatility, but high risk also equates to high returns. Therefore, it is recommended that different investors adopt different investment strategies, for example, for risk-averse or neutral investors, it is suitable to invest under the second risk price; for risk-loving investors, it is suitable to invest under the first risk price.And we find that the CEV-SV Model degenerates to SV Model when $\beta = 0$,which means the presence of CEV component incites a sharp downward movements in the optimal allocation toward short maturities.

Our analysis can be extended to solve the investment problems in the presence of transaction costs,stochastic affine interest rates,and the other uncertain factors,which will involve more complicated HJB equations to solve. We leave this work for a future study.

V. ACKNOWLEDGEMENTS

Thanks to the National Natural Science Foundation of China for funding the project (12061066); Funding support from the Natural Science Foundation of Gansu Province (20JR5RA528).

APPENDIX

- Organize (17) by v yields

$$0 = h_t + \kappa\theta h_v + v[-\kappa h_v + \frac{1}{2}\sigma^2 h_{vv} - \frac{1}{2}\sigma^2 q(1-\rho^2)h_v^2 - \sigma\rho\bar{\lambda}h_v + \frac{1}{2q}\bar{\lambda}^2] \quad (A.1)$$

Further, we can decompose (A.1) into two equations

$$h_t + \kappa\theta h_v = 0 \quad (A.2)$$

$$-\kappa h_v + \frac{1}{2}\sigma^2 h_{vv} - \frac{1}{2}\sigma^2 q(1-\rho^2)h_v^2 - \sigma\rho\bar{\lambda}h_v + \frac{1}{2q}\bar{\lambda}^2 = 0 \quad (A.3)$$

The value obtained from the second derivative of function h can be ignored. Therefore, for the convenience of calculation, here we assume $\frac{1}{2}\sigma^2 h_{vv} = 0$.

Thereby, we here

$$\frac{1}{2}\sigma^2 q(1-\rho^2)h_v^2 + (\kappa + \sigma\rho\bar{\lambda})h_v - \frac{1}{2q}\bar{\lambda}^2 = 0 \quad (A.4)$$

This is a quadratic equation about h_v ,and the solution is

$$h_v = \frac{-(\kappa+\sigma\rho\bar{\lambda}) \pm \sqrt{(\kappa+\sigma\rho\bar{\lambda})^2 + \sigma^2 q^2(1-\rho^2)\bar{\lambda}^2}}{\sigma^2 q(1-\rho^2)} := K_1 \quad (A.5)$$

With $h(T,v) = 0$, we can get

$$h(t,v) = K_1\kappa\theta(T-t) \quad (A.6)$$

$$\psi^* = \frac{\bar{\lambda}\sqrt{v}}{e^{r(T-t)}xq} \quad (A.7)$$

As (7),we get

$$\pi^* = \frac{\psi^*}{S^{\square}(a\sqrt{v} + \frac{b}{\sqrt{v}})} = \frac{\bar{\lambda}\sqrt{v}}{S^{\beta}(a\sqrt{v} + \frac{b}{\sqrt{v}})e^{r(T-t)}xq} \quad (A.8)$$

- Organize (17) by v and $\sqrt{v}$ yields

$$0 = \frac{1}{2q}\bar{\lambda}_c^{\,2} + h_t + \kappa\theta h_v + v[-\kappa h_v + \frac{1}{2}\sigma^2 h_{vv} - \frac{1}{2}\sigma^2 q(1-\rho^2)h_v^2] - \sigma\rho\sqrt{v}\bar{\lambda}_c h_v \quad (A.9)$$

Further, we can decompose (A.9) into three equations

$$\frac{1}{2q}\bar{\lambda}_c^{\,2} + h_t + \kappa\theta h_v = 0$$

$$-\kappa h_v + \frac{1}{2}\sigma^2 h_{vv} - \frac{1}{2}\sigma^2 q(1-\rho^2)h_v^2 = 0 \quad (A.10)$$

$$\sigma\rho\sqrt{v}\bar{\lambda}_c h_v = 0$$

With $h(T,v) = 0$, we can get

$$h(t,v) = \frac{1}{2q}\bar{\lambda}_c^{\,2}(T-t) \quad (A.11)$$

$$\psi^* = \frac{\bar{\lambda}_c}{e^{r(T-t)}xq} \quad (A.12)$$

As (7), we get

$$\pi^* = \frac{\psi^*}{S^{\beta}(a\sqrt{v} + \frac{b}{\sqrt{v}})} = \frac{\bar{\lambda}_c}{S^{\beta}(a\sqrt{v} + \frac{b}{\sqrt{v}})e^{r(T-t)}xq} \quad (A.13)$$

REFERENCES

[1] Merton, Robert C.Lifetime portfolio selection under uncertainty: The continuous-time case. The review of Economics and Statistics 1969, 51, 247–57.
[2] Cox, John. Constant Elasticity of Variance Diffusions. Notes on Option Pricing I 1975, Unpublished note, Stanford University,Graduate School of Business, Stanford, CA, USA.
[3] Cox, John C., and Stephen A. Ross. The valuation of options for alternative stochastic processes. Journal of Financial Economics 1976 ,3,145–66.
[4] Beckers, Stan. The constant elasticity of variance model and its implications for option pricing.The Journal of Finance 1980, 35, 661–73.
[5] MacBeth, James D., and Larry J. Merville. Tests of the black-scholes and cox call option valuation models. The Journal of Finance 1980 , 35, 285–301.
[6] Emanuel, David C., and James D. MacBeth. Further results on the constant elasticity of variance call option pricing model. Journal of Financial and Quantitative Analysis 1982 , 17,533–54.
[7] Cox, John C.The constant elasticity of variance option pricing model. Journal of Portfolio Management 1996 , 23,15–17.
[8] Davydov, Dmitry, and Vadim Linetsky. 2001. The valuation and hedging of barrier and lookback options under the cev process.Management Science 2001 , 47 , 949–65.
[9] Lo, Chi Fai, P. H. Yuen, and Cho-Hoi Hui.Constant elasticity of variance option pricing model with time-dependent parameters.International Journal of Theoretical and Applied Finance 2000 , 3, 661–74.
[10] Gao, JW (Gao, Jianwei).Optimal portfolios for DC pension plans under a CEV model.Insurance: Mathematics and Economics 2009 , 3, 479-490.
[11] Ghysels, Eric, Andrew C. Harvey, and Eric Renault. 5 stochastic volatility. Handbook of Statistics 1996,14: 119-91.
[12] Harvey, Campbell R.. The specification of conditional expectations. Journal of Empirical Finance2001, 8: 573-637
[13] Kim, Jeong-Hoon, Jungwoo Lee, Song-Ping Zhu, and Seok-Hyon Yu. A multiscale correction to the black-scholes formula.Applied Stochastic Models in Business and Industry 2014, 36: 753-65.
[14] Escobar-Anel, Marcos;Fan, Weili. The SEV-SV Model—Applications in Portfolio Optimization. Risks 2023 , 230.
[15] Grasselli, Martino. The 4/2 stochastic volatility model: A unified approach for the heston and the 3/2 model. Mathematical Finance 2017 , 27, 1013–34.
[16] Heston, Steven L. A closed-form solution for options with stochastic volatility with applications to bond and currency options. The Review of Financial Studies 1993 , 6, 327–43.
[17] Heston, Steven L. A Simple New Formula for Options with Stochastic Volatility. St. Louis: Washington University in St. Louis,1997 .
[18] Platen, E. 1997. A Non-Linear Stochastic Volatility Model. Financial Mathematics Research Report No. FMRR 005-97. Canberra: Center for Financial Mathematics, Australian National Universite.
[19] Cui, Zhenyu, J. Lars Kirkby, and Duy Nguyen.A general framework for discretely sampled realized variance derivatives in stochastic volatility models with jumps. European Journal of Operational Research 2017 , 262,381–400.
[20] Cui, Zhenyu, J. Lars Kirkby, and Duy Nguyen.A General Valuation Framework for SABR and Stochastic Local Volatility Models SIAM Journal on Financial Mathematics 2018 , 9, 520–63.
[21] Chacko, George and Viceira, Luis M. Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Markets. The Review of Financial Studies 2005, 18, 1369-1402.
[22] Boguslavskaya, Elena, and Dmitry Muravey. An explicit solution for optimal investment in heston model. Theory of Probability AND Its Applications 2016, 60, 679–88.
[23] Cheng, Yuyang;Escobar-Anel, Marcos. Optimal investment strategy in the family of 4/2 stochastic volatility models. Quantitative Finance 2021, 21, 1723–51.
[24] Gao, Jianwei. An extended cev model and the legendre transform–dual–asymptotic solutions for annuity contracts. Insurance:Mathematics and Economics 2010 , 46 511–30.

[25] Jiling Cao and Jeong-Hoon Kim and Wenjun Zhang. Pricing variance swaps under hybrid CEV and stochastic volatility. Journal of Computational and Applied Mathematics 2021, 386, 113220.
[26] Choi, Sun-Yong, Jean-Pierre Fouque, and Jeong-Hoon Kim.Option pricing under hybrid stochastic and local volatility. Quantitative Finance 2013 , 13 , 1157–65.
[27] R.C. Merton. Optimum consumption and portfolio rules in a continuoustime model. Journal of Economic Theory 1971 , 3 373-413.
[28] Jean-Luc Vila and Thaleia Zariphopoulou. Optimal Consumption and Portfolio Choice with Borrowing Constraints. Journal of Economic Theory 1997 , 77 0022-0531.
[29] Eun Ju Jung and Jai Heui Kim. Optimal investment strategies for the HARA utility under the constant elasticity of variance model. Insurance: Mathematics and Economics 2012 , 51 667-673.
[30] He Yong;Chen Peimin;He Lin;Xiang Kaili;Wu Chunchi. A dynamic Heston local–stochastic volatility model and Legendre transform dualasymptotic solution for optimal investment strategy problems with CARA utility. J. Comput. Appl. Math. 2023 ,114993.