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High-Dimensional Intrinsic Open Quantum Walks

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Abstract— **As a quantum analogue of classical Markov chains, open quantum walks are widely used in many research fields such as quantum computing and quantum simulation. In this paper, we extend the intrinsic open quantum walk model defined on one-dimensional integer lattice to higher-dimensional integer lattices and study its structural properties.**

***Index Terms*—Open Quantum walk, Quantum channel, Probability distribution, Integer lattices.**

I. Introduction

As a quantum analogue of classical random walks, quantum walks[1] have a wide range of applications in fields such as quantum information[2], quantum computing[3] and biology[4]. Quantum walks are mainly divided into discrete-time quantum walks and continuous-time quantum walks. According to whether they interact with the environment or not, quantum walks can be divided into two categories: unitary quantum walks and open quantum walks[5]. In fact, every quantum system is actually open. However, closed quantum system is only a theoretical desirable model. An open quantum walk is only a kind of non-unitary dynamical model describing open quantum system, which was first introduced by Attal[6] et al. in 2012. The central limit theorem for the open quantum walks on -dimensional integer lattice was proved in [7]. In this paper, we will focus on open quantum walks on the -dimensional integer lattice .

With the in-depth study of quantum walks, we gradually realize that the extension of quantum walks to higher dimensions (i.e., high-dimensional quantum walks) can reveal more complicated quantum phenomena and have great impact on quantum computation and quantum information processing. In 2002, Mackay and Bartlett et al.[8] extended Hadamard walk to the high-dimensional case and examined the time dependence of standard deviation, which revealed a general characterization of the secondary gain of the classical random walks. Szegedy[9] introduced a general method for quantizing classical algorithms based on random walks and generalized the celebrated result of Ambainiset. that computes properties of quantum walks on the -dimensional torus. In 2017,Komatsu and Konno[10] studied the steady-state amplitude of quantum walks on high-dimensional integer lattice, obtaining a smooth measure for Grover's walks.

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In recent years, open quantum walks on high-dimensional integer lattices have attracted increasing attention from experts and have made remarkable progress, including the establishment of theoretical models and breakthroughs in experimental verification, which lays groundwork for the application prospect of high-dimensional quantum walks. In 2020, Wang [11] et al. proposed a discrete-time quantum walk model on a -dimensional integer lattice using quantum Bernoulli noises. In 2023, Esposito[12] et al. used two-dimensional quantum oscillatory evolution as a tool for generating high-dimensional quantum states. And they succeeded in generating high-dimensional quantum states using two-dimensional quantum walks on an experimental photonics platform. In the same year, Wang and Zhan et al. [13] realized a general positive-operator valued measurement on a high-dimensional quantum system, and using single photon and linear optics, a general positive-operator valued measurement was experimentally achieved on a three-dimensional system with high fidelity.

In this paper, we extend the intrinsic open quantum walks on the integer lattice to a higher dimensional case. More precisely, for a general positive integer , we will introduce a high-dimensional model of intrinsic open quantum walk on a -dimensional integer lattice and study its constructive properties.

II. Preliminaries

In this section, we recall some necessary notions and facts about Hilbert.

Assume $d \geq 2$ is a given positive integer. We denote by $¢^{d}$ the set of d-weighted Cartesian products of the set of integers ¢ , namely

$$¢^{d} = \{x = (x_1, x_2, \cdots, x_d) \mid x_1, x_2, \cdots, x_d \in ¢\}. \quad (1)$$

In the literature, $¢^{d}$ is often referred to as the d-dimensional integer lattice.

The model of open quantum walks on a d-dimensional integer lattice $¢^{d}$ are aimed at describing the walking behavior of a quantum particle with 2^d degrees of freedom on $¢^{d}$. To describe this process, it involves two spaces: one is called the position space, which describes the positional state of the quantum particle, the other is called the coin space, which describes the internal degrees of freedom of the quantum particle. Their tensor product space is used to describe the state of the quantum particle. This is called the state space. Let $£^{d}$ be coin space, this section briefly introduces some concepts, notations, and facts about the position space and its tensor product space with $£^{d}$.

Let $l^2(¢^{d})$ be the space of square summable functions on the d-dimensional lattice $¢^d$, namely

$$l^2(¢^d) = \{\xi : ¢^d \to £ \mid \sum_{x\in ¢^d} |\xi(x)|^2 < \infty\}, \quad (2)$$

with the natural linear operations and the inner product $\langle\cdot,\cdot\rangle$ given by

$$\langle \xi,\eta\rangle = \sum_{x\in ¢^d} \overline{\xi(x)}\eta(x), \quad \xi,\eta \in l^2(¢^d),$$

where $\overline{\xi(x)}$ denotes complex conjugate. As a separable complex Hilbert space, $l^2(¢^d)$ has an orthonormal basis (ONB) of the form $\{\delta_z \mid z\in ¢^d\}$, where δ_z is the Dirac delta function on $¢^d$ defined by

$$\delta_z(x) = \begin{cases} 1, x = z;\ x\in ¢^d; \\ 0, x \neq z;\ x\in ¢^d. \end{cases}$$

We call $\{\delta_z \mid z\in ¢^d\}$ the canonical ONB for $l^2(¢^d)$.

For $\xi \in l^2(¢^d)$, it is easy to see that $\langle \delta_z, \xi\rangle_{l^2(¢^d)} = \xi(z)$, $z\in ¢^d$, thus, each $\xi\in l^2(¢^d)$ has a Fourier expansion of the form

$$\xi = \sum_{z\in ¢^d} \langle \delta_z,\xi\rangle_{l^2(¢^d)}\delta_z = \sum_{z\in ¢^d} \xi(z)\delta_z,$$

where the series on righthand side converges in norm $\|\cdot\|_{l^2(¢^d)} = \sqrt{\langle\cdot,\cdot\rangle_{l^2(¢^d)}}$.

We introduce the notation $\Lambda = \{-1,+1\}$, and denote by Λ^d the set of d-weighted Cartesian product of Λ, namely $\Lambda^d = \{\varepsilon = (\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_d) \mid \varepsilon_j \in \Lambda, 1\le j\le d\}$.

It is easy to see that $\Lambda^d \subseteq ¢^d$, and $\#\Lambda^d = 2^d$, where $\#\Lambda^d$ means the cardinality of Λ^d, namely Λ^d is a 2^d-element subset of the d-dimensional integer lattice $¢^d$. Furthermore, it can be shown that if $-\varepsilon = (-\varepsilon_1,-\varepsilon_2,\cdots,-\varepsilon_d)$, then $-\varepsilon\in\Lambda^d$.

Given $\varepsilon\in\Lambda^d$ and the function $f: ¢^d \to £$, we denote by f_ε

$$f_\varepsilon(x) = f(x+\varepsilon), \quad x\in ¢^d. \quad (3)$$

It is easy to see that if $f\in l^2(¢^d)$, $f_\varepsilon \in l^2(¢^d)$, moreover $\|f\|_{l^2(¢^d)} = \|f_\varepsilon\|_{l^2(¢^d)}$.

Definition 2.1. Let $\varepsilon\in\Lambda^d$ be gived, there exist the ε-shift operators S_ε on $l^2(¢^d)$ such that $S_\varepsilon f = f_\varepsilon, f\in l^2(¢^d)$.

Lemma 2.1. Let $\varepsilon\in\Lambda^d$ be give, then the ε-shift operator S_ε is a unitary operator on $l^2(¢^d)$, whose adjoint operator is $-\varepsilon$-shift operator $S_{-\varepsilon}$, such that $S_\varepsilon^* = S_{-\varepsilon}$.

Let $f,g\in l^2(¢^d)$ be gived, the Dirac operator denote by $|f\rangle\langle g|$, namely $|f\rangle\langle g|$ is defined by the following

$$|f\rangle\langle g| u = \langle g,u\rangle_{l^2(¢^d)} f, \quad u\in l^2(¢^d). \quad (4)$$

It is easy to see that $|f\rangle\langle g|$ is a bounded operator on $l^2(¢^d)$, In particular, if $\|f\|_{l^2(z^d)} = 1$, then $|f\rangle\langle g|$ is a one-dimensional projection operator on $l^2(¢^d)$.

According to the general theory of functional analysis, the projection operator system $\{\ |\delta_x\rangle\langle\delta_x|\ \mid x\in ¢^d\ \}$ constructed by the canonical ONB $\{\delta_x \mid x\in ¢^d\}$ for $l^2(¢^d)$ has partition of identity. Namely, for $z_1, z_2 \in ¢^d$ with $z_1 \neq z_2$, we have $|\delta_{z_1}\rangle\langle\delta_{z_1}||\delta_{z_2}\rangle\langle\delta_{z_2}| = 0$ and

$$I_{l^2(¢^d)} = \sum_{z\in ¢^d} |\delta_z\rangle\langle\delta_z|. \quad (5)$$

Here $I_{l^2(¢^d)}$ is the identity operator on $l^2(¢^d)$, where the operator series converge strongly.

Definition 2.2. Let $\mathcal{U} = \{A_\varepsilon \mid \varepsilon\in\Lambda^d\}$ be a set of bounded operators on the space $£^d$ with index set Λ^d, and satisfies the following conditions, $\sum_{\varepsilon\in\Lambda^d} A_\varepsilon$ is unitary operator on $£^d$;

For $\varepsilon,\varepsilon'\in\Lambda^d$ with $\varepsilon = \varepsilon'$, we have $A_\varepsilon^* A_{\varepsilon'} = A_\varepsilon A_{\varepsilon'}^* = 0$.

We call $\mathcal{U} = \{A_\varepsilon \mid \varepsilon\in\Lambda^d\}$ the Λ^d-coin operator system on $£^d$.

It is easy to see that the notion of Λ^d-coin operator system is a natural generalization of the notion of coin operator pair in the open quantum walk on $¢^d$. The following proposition describes the operation property of the Λ^d-coin operator system.

Proposition 2.1. Let $\mathcal{U} = \{A_\varepsilon \mid \varepsilon\in\Lambda^d\}$ be a Λ^d-coin operator system on $£^d$ and $I_{£^d}$ is a family identity operators on $£^d$. Then we have

$$\sum_{\varepsilon\in\Lambda^d} A_\varepsilon^* A_\varepsilon = \sum_{\varepsilon\in\Lambda^d} A_\varepsilon A_\varepsilon^* = I_d. \quad (6)$$

Where I_d is the identity operator on $£^d$.

Moerover, for each $\varepsilon\in\Lambda^d$, the product $A_\varepsilon^* A_\varepsilon$ and $A_\varepsilon A_\varepsilon^*$ are projection operators on $£^d$.

Proof. Let $U = \sum_{\varepsilon\in\Lambda^d} A_\varepsilon$, then $U^* = \sum_{\varepsilon\in\Lambda^d} A_\varepsilon^*$. Thus,

$$I_d = U^*U = (\sum_{\varepsilon\in\Lambda^d} A_\varepsilon^*)(\sum_{\varepsilon\in\Lambda^d} A_\varepsilon) = \sum_{\varepsilon\in\Lambda^d}\sum_{\varepsilon'\in\Lambda^d} A_\varepsilon^* A_{\varepsilon'}.$$

On the other hand, when $\varepsilon, \varepsilon' \in \Lambda^d$ and $\varepsilon \neq \varepsilon'$, we have $A_\varepsilon^* A_{\varepsilon'} = 0$. Applying this property to the above summation yields

$$\sum_{\varepsilon\in\Lambda^d} A_\varepsilon^* A_\varepsilon = I.$$

Similarly, we can verify the equation $\sum_{\varepsilon\in\Lambda^d} A_\varepsilon A_\varepsilon^* = I$.

In the following, we prove the remaining conclusions. Setting $\varepsilon\in\Lambda^d$. Clearly $A_\varepsilon^* A_\varepsilon$ and $A_\varepsilon A_\varepsilon^*$ are self-adjoint operators. Using the equation $\sum_{\varepsilon\in\Lambda^d} A_\varepsilon^* A_\varepsilon = I$, we have

$$(A_\varepsilon^* A_\varepsilon)^2 = A_\varepsilon^* A_\varepsilon (I - \sum_{\varepsilon'\in\Lambda^d, \varepsilon'\neq\varepsilon} A_{\varepsilon'}^* A_{\varepsilon'}) =$$

$$A_\varepsilon^* A_\varepsilon - \sum_{\varepsilon'\in\Lambda^d, \varepsilon'\neq\varepsilon} A_\varepsilon^* A_\varepsilon A_{\varepsilon'}^* A_{\varepsilon'} = A_\varepsilon^* A_\varepsilon.$$

So $A_\varepsilon^* A_\varepsilon$ is a projection operator. Similarly, $A_\varepsilon A_\varepsilon^*$ is also a projection operator.

The natural choice of state space is the tensor space $l^2(¢^d)\otimes £^d$. The following are some facts about $l^2(¢^d)\otimes £^d$. We respectively use $\langle \cdot,\cdot\rangle$ and $\|\cdot\|$ to represent the inner product and norm in $l^2(¢^d)\otimes £^d$.

Lemma 2.2. Let $l^2(¢^d)\otimes £^d$ be the tensor product space, it admits the following features:

For $\xi_1, \xi_2 \in l^2(¢^d),\ u_1, u_2 \in £^d$, we get

$$\langle \xi_1\otimes u_1, \xi_2\otimes u_2\rangle = \langle \xi_1, \xi_2\rangle_{l^2(¢^d)} \langle u_1, u_2\rangle_{£^d}, \quad (7)$$

where $\langle \cdot,\cdot\rangle_{£^d}$ is the inner product on $£^d$;

$\overline{span}\ \{\xi\otimes u \mid \xi\in l^2(¢^d), u\in £^d\}$ is a complete subset of $l^2(¢^d)\otimes £^d$;

Let A and B be bounded operators on $l^2(¢^d)$ and $£^d$, respectively. Then $A\otimes B$ is bounded operator on $l^2(¢^d)\otimes £^d$ and satisfies

$$A\otimes B(\xi\otimes u) = (A\xi)\otimes(Bu), \xi\in l^2(¢^d), u\in £^d. \quad (8)$$

moreover, $\|A\otimes B\| = \|A\|\ \|B\|$.

The Banach space consisting of all trace class operators on the tensor product space $l^2(¢^d)\otimes £^d$ is denoted by $\mathcal{S}(l^2(¢^d)\otimes £^d)$ with norm $\|\cdot\|_1$. In addition, let $\mathcal{S}_+(l^2(¢^d)\otimes £^d)$ be positive elements on $\mathcal{S}(l^2(¢^d)\otimes £^d)$. That is

$$\mathcal{S}_+(l^2(¢^d)\otimes £^d) = \{\hat{\rho}\in\mathcal{S}(l^2(¢^d)\otimes £^d) \mid \hat{\rho}\geq 0\}. \quad (9)$$

If $\hat{\rho}\in\mathcal{S}_+(l^2(¢^d)\otimes £^d)$ and $\mathrm{Tr}\,\hat{\rho}=1$, then $\hat{\rho}$ is a density operator on $l^2(¢^d)\otimes £^d$. The set of all density operators on space $l^2(¢^d)\otimes £^d$ is denoted by $\mathcal{D}(l^2(¢^d)\otimes £^d)$.

Similar to the tensor product space, we denote by $\mathcal{S}_+(£^d)$ the set of all positive trace class operators on $£^d$. In fact, $\mathcal{S}_+(£^d)$ is the set of all positive operators on $£^d$. In the following, we denote by $\mathrm{Tr}A$ the trace of operator A.

Definition 2.3. There is a map $\rho: ¢^d \to \mathcal{S}_+(£^d)$, it satisfies $\sum_{x\in¢^d} \mathrm{Tr}[\rho(x)] = 1$. We refer to ρ as d-dimension nucleus from $¢^d$ to $£^d$. The set of all d-dimension nucleus from $¢^d$ to $S_+(£^d)$ is denoted by $\mathrm{Nuc}(¢^d, £^d)$.

The following lemma shows that density operators on the tensor product space $l^2(¢^d)\otimes £^d$ can be constructed by nucleus from $¢^d$ to $S_+(£^d)$.

Lemma 2.3. Let $\rho\in\mathrm{Nuc}(¢^d, £^d)$ be given. Then for each $x\in¢^d$, the corresponding tensor product operator $|\delta_x\rangle\langle\delta_x|\otimes\rho(x)$ is a positive trace class operator on $l^2(¢^d)\otimes £^d$. Moreover, the operator series

$$\sum_{x\in¢^d} |\delta_x\rangle\langle\delta_x|\otimes\rho(x) \quad (10)$$

converge in norm of trace operator space $\mathcal{S}(l^2(¢^d)\otimes £^d)$, and its sum operator is the density operator on the space $l^2(¢^d)\otimes £^d$.

Thus, the sum of operator series (10) belongs to $\mathcal{D}(l^2(¢^d)\otimes £^d)$. In the following, we use $\mathcal{D}_{\mathrm{Nuc}}(l^2(¢^d)\otimes £^d))$ to denote the set of operator series, namly

$$\mathcal{D}_{\mathrm{Nuc}}(l^2(¢^d)\otimes £^d) = \{\hat{\rho} \mid \hat{\rho} = \sum_{x\in¢^d} |\delta_x\rangle\langle\delta_x|\otimes\rho(x),$$

$$\rho\in\mathrm{Nuc}(¢^d, £^d)\}. \quad (11)$$

The element in $\mathcal{D}_{\mathrm{Nuc}}(l^2(¢^d)\otimes £^d)$ is a density operator with a Nucleus on $l^2(¢^d)\otimes £^d$.

The intrinsic open quantum walk model that we introduce on the one-dimensional integer lattice is given below.

Lemma 2.4. Let $\{C,D\}$ be the coin operator pair of $£^2$. Then the one-dimensional integer lattice $\mathbf{Z}$ generated by $\{C,D\}$ generated by intrinsic open quantum walk is a quantum dynamical model with the following features:

The walk takes $l^2(¢)\otimes £^2$ as its state space and its states are represented by unit vectors in $l^2(¢)\otimes £^2$;

The evolution of its state is governed by equation

$$\omega^{(n)} = \mathrm{L}^n\left(\omega^{(0)}\right), \quad n \geq 0, \omega^{(n)} = \mathrm{L}^n\left(\omega^{(0)}\right), \quad n \geq 0, \tag{12}$$

where L is the intrinsic quantum channel generated by $\{C,D\}$, L^n denotes its n-th composition. (L^0 is defined as the corresponding identity mapping), and $\omega^{(n)}$ is the state of the walk at time n, in particular $\omega^{(0)}$ is the initial state.

III. Main Results

In this section, we state and prove our main results.

As mentioned above, in open quantum walk model on the d-dimensional integer lattice $¢^d$, we use $l^2(¢^d)$ and $£^d$ to represent the position space and coin space, respectively.

In the following, $I_{£^d}$, $I_{l^2(¢^d)}$ and I denote dentity operator on $£^d$, $l^2(¢^d)$ and $l^2(¢^d)\otimes £^d$ respectiveiy.

Definition 3.1. Let $\mathcal{U} = \{A_\varepsilon \mid \varepsilon \in \Lambda^d\}$ be a system of bounded operators on $£^d$ with index set Λ^d. If

$$\sum_{\varepsilon\in\Lambda^d} A_\varepsilon^* A_\varepsilon = I_{£^d}. \tag{13}$$

then $\mathcal{U} = \{A_\varepsilon \mid \varepsilon \in \Lambda^d\}$ is said to be a Λ^d-generalized operator system on $£^d$.

It is easy to see that the Λ^d-coin operator system on $£^d$ must be a Λ^d-generalized coin operator system, but the converse does not necessarily hold. In the literature, generalized coin operator system are also known as Kraus operator system. The following proposition shows a specific application of generalized coin operator system.

Proposition 3.1. Let $\mathcal{U} = \{A_\varepsilon \mid \varepsilon \in \Lambda^d\}$ be a Λ^d-generalized coin operator system on $£^d$. If $\rho \in \mathcal{S}_+(£^d)$, then $\sum_{\varepsilon\in\Lambda^d} A_\varepsilon \rho A_\varepsilon^* \in \mathcal{S}_+(£^d)$, and $\mathrm{Tr}[\sum_{\varepsilon\in\Lambda^d} A_\varepsilon \rho A_\varepsilon^*] = \mathrm{Tr}\rho$.

Proof. For each indicator $\varepsilon \in \Lambda^d$, it is easy to see $A_\varepsilon \rho A_\varepsilon^*$ is a positive operator, according to the properties of the trace class operator, $A_\varepsilon \rho A_\varepsilon^*$ is still a trace class operator. Thus $A_\varepsilon \rho A_\varepsilon^* \in \mathcal{S}_+(£^d)$, this mean $\sum_{\varepsilon\in\Lambda^d} A_\varepsilon \rho A_\varepsilon^* \in \mathcal{S}_+(£^d)$.

Applying the properties of the trace class operator and $\sum_{\varepsilon\in\Lambda^d} A_\varepsilon A_\varepsilon^* = I_{£^d}$, we get

$$\mathrm{Tr}[\sum_{\varepsilon\in\Lambda^d} \mathrm{A}_\varepsilon \rho \mathrm{A}_\varepsilon^*] = \sum_{\varepsilon\in\Lambda^d} \mathrm{Tr}[\mathrm{A}_\varepsilon \rho \mathrm{A}_\varepsilon^*] = \sum_{\varepsilon\in\Lambda^d} \mathrm{Tr}[\rho \mathrm{A}_\varepsilon \mathrm{A}_\varepsilon^*]$$
$$= \mathrm{Tr}[\rho \sum_{\varepsilon\in\Lambda^d} \mathrm{A}_\varepsilon \mathrm{A}_\varepsilon^*] = \mathrm{Tr}\rho.$$

Lemma 3.1. Let $\varepsilon \in \Lambda^d$, $\{\delta_z \mid z \in ¢^d\}$ be the ONB on $l^2(\mathbf{Z}^d)$. Then, the ε-drift operator S_ε has an operator series representation of the following form

$$S_\varepsilon = \sum_{z\in ¢^d} |\delta_z\rangle\langle\delta_{z+\varepsilon}|, \tag{14}$$

where the operator series on righthand side converges strongly. In particular, S_ε satisfies $S_\varepsilon \delta_z = \delta_{z-\varepsilon}, \forall z \in ¢^d$.

According to Lemma 2.1., it can be inferred that $S_\varepsilon^* = S_{-\varepsilon}$, namely $S_{-\varepsilon}\delta_z = \delta_{z+\varepsilon}, \forall z \in ¢^d$.

Let $\mathcal{U} = \{A_\varepsilon | \varepsilon \in \Lambda^d\}$ be a Λ^d-generalized coin operator system on $£^d$. A bounded operator $\mathrm{L}^{\mathcal{U}}$ on $l^2(¢^d)\otimes £^d$ can be defined as follows

$$\mathrm{L}^{(\mathcal{A})} = S_\varepsilon \otimes A_\varepsilon, \tag{15}$$

where $\varepsilon \in \Lambda^d$. Then the system of operators $\{\mathrm{L}^{\mathcal{U}}\}$ is said to be Kraus operators system generated by $\mathcal{U}$ in the Unitstensor product space $l^2(¢^d)\otimes £^d$, and the elements of this system are called the Kraus operators generated by $\mathcal{U}$.

Theorem 3.1. Let $\mathcal{U} = \{A_\varepsilon | \varepsilon \in \Lambda^d\}$ be a Λ^d-generalized coin operator system on $£^d$. Then the Kraus operator system $\{\mathrm{L}^{\mathcal{U}}\}$ has the following properties

$$\mathrm{L}^{*(\mathcal{U})}\mathrm{L}^{(\mathcal{U})} = I. \tag{16}$$

Proof. Using the properties of the tensor product operator, we obtain

$$\begin{aligned}
\mathrm{L}^{(\mathcal{U})^*}\mathrm{L}^{(\mathcal{U})} &= (S_\varepsilon \otimes A_\varepsilon)^*(S_\varepsilon \otimes A_\varepsilon) \\
&= (S_{-\varepsilon} \otimes A_\varepsilon^*)(S_\varepsilon \otimes A_\varepsilon) \\
&= (S_{-\varepsilon}S_\varepsilon) \otimes (A_\varepsilon^* A_\varepsilon) \\
&= I_{l^2(¢^d)} \otimes I_{£^d} \\
&= I
\end{aligned}.$$

where $I_{l^2(¢^d)}$ and $I_{£^d}$ denote the identity operator on $l^2(¢^d)$ and $£^d$ respectively. The proposition is proved.

Let $\mathcal{U}=\{A_\varepsilon|\varepsilon\in\Lambda^d\}$ be a Λ^d -generalized coin operator system on $£^d$. The mapping $\mathrm{L}_{(\mathcal{U})}$ on $\mathcal{S}(l^2(¢^d)\otimes £^d)$ be defined as follows

$$\mathrm{L}_{(\mathcal{U})}(\omega)=\mathrm{L}^{(\mathcal{U})}\omega\mathrm{L}^{(\mathcal{U})^*},\quad \omega\in\mathcal{S}(l^2(\mathbf{Z}^d)\otimes £^d). \tag{17}$$

By Theorem 3.1, we can see that it satisfies $\mathrm{Tr}[\mathrm{L}^{(\mathcal{U})}(\omega)]=\mathrm{Tr}\,\omega$, $\forall\omega\in\mathcal{S}(l^2(¢^d)\otimes £^d)$, and $\mathcal{S}_+(l^2(¢^d)\otimes £^d)$ is invariant under the action of $\mathrm{L}_{(\mathcal{U})}$. Thus

$$\mathrm{L}_{(\mathcal{U})}(\omega)\in\mathcal{S}(l^2(¢)\otimes £^d),\quad \forall\omega\in\mathcal{S}(l^2(¢)\otimes £^d). \tag{18}$$

The map $\mathrm{L}_{(\mathcal{U})}$ is a d -dimensional quantum channel on $l^2(¢^d)\otimes £^d$.

Let $\mathcal{U}=\{A_\varepsilon\,|\,\varepsilon\in\Lambda^d\}$ be a Λ^d -generalized coin operators system on $£^d$. Then the quantum channel $\mathrm{L}_{(\mathcal{U})}$ defined by (17) is called the d -dimensional intrinsic quantum channel generated by Λ^d -generalized coin operator system $\mathcal{U}$ on $l^2(¢^d)\otimes £^d$. The d -dimensional intrinsic quantum channel generated by $\mathcal{U}$, abbreviated as $\mathcal{U}$ -generated d -dimensional intrinsic quantum channel.

It is easy to see that the d -dimensional intrinsic quantum channel $\mathrm{L}_{(\mathcal{U})}$ generated by $\mathcal{U}$ is independent of the ONB in $l^2(¢^d)$, and it is an extension of the intrinsic quantum channel on the integer lattice $¢$ .The next theorem further gives analytic and algebraic properties of the d -dimensional intrinsic quantum channel $\mathrm{L}_{(\mathcal{U})}$.

Theorem 3.2. Let $\mathcal{U}=\{A_\varepsilon\,|\,\varepsilon\in\Lambda^d\}$ be a Λ^d-generalized coin operators system on the space $£^d$, $\mathrm{L}_{(\mathcal{U})}$ is the intrinsic quantum channel generated by $\mathcal{U}$. Then $\mathrm{L}_{(\mathcal{U})}$ is a continuous linear mapping in the Banach space $\mathcal{S}(l^2(¢^d)\otimes £^d)$, and $\mathcal{D}_{\text{Nuc}}(l^2(¢^d)\otimes £^d)$ is invariant under the action of $\mathrm{L}_{(\mathcal{U})}$, namely one has

$$\mathrm{L}_{(\mathcal{U})}(\omega)\in\mathcal{D}_{\text{Nuc}}\left(l^2(¢^d)\otimes £^d\right),\quad \forall\omega\in\mathcal{D}_{\text{Nuc}}\left(l^2(¢^d)\otimes £^d\right). \tag{19}$$

Proof. Clearly $\mathrm{L}_{(\mathcal{U})}$ is a linear mapping on the Banach space $\mathcal{S}(l^2(¢^d)\otimes £^d)$. Let $\omega\in\mathcal{S}(l^2(¢^d)\otimes £^d)$. From the relation between the operator norm and trace norm of the operator it follows that

$$\begin{aligned}\|\mathrm{L}_{(\mathcal{U})}(\omega)\|_1&=\|(S_\varepsilon\otimes A_\varepsilon)\omega(S_\varepsilon\otimes A_\varepsilon)^*\|_1\\&\le\|S_\varepsilon\otimes A_\varepsilon\|\ \|\omega\|_1\|(S_\varepsilon\otimes A_\varepsilon)^*\|\\&=\|S_\varepsilon\otimes A_\varepsilon\|^2.\end{aligned}$$

This shows that $\mathrm{L}_{(\mathcal{U})}$ is also bounded on the Banach space $\mathcal{S}(l^2(¢)\otimes £^d)$. Thus $\mathrm{L}_{(\mathcal{U})}$ is a continuous linear mapping on the Banach space $\mathcal{S}(l^2(¢)\otimes £^d)$. For $\omega\in\mathcal{D}_{\text{Nuc}}\left(l^2(¢)\otimes £^d\right)$, we have

$$\omega=\sum_{x\in¢^d}|\delta_x\rangle\langle\delta_x|\otimes\rho(x),\rho\in\mathrm{Nuc}\left(¢^d,£^d\right).$$

We define $\tilde{\rho}(x)=A_\varepsilon\rho(x)A_\varepsilon^*$, it has that $\tilde{\rho}\in\mathrm{Nuc}\left(¢^d,£^d\right)$. For each $x\in¢$, using the properties of the shift operator, we get

$$\begin{aligned}&\mathrm{L}_{(\mathcal{U})}(|\delta_x\rangle\langle\delta_x|\otimes\rho(x))\\&=(S_\varepsilon\otimes A_\varepsilon)(|\delta_x\rangle\langle\delta_x|\otimes\rho(x))(S_\varepsilon\otimes A_\varepsilon)^*\\&=(S_\varepsilon|\delta_x\rangle\langle\delta_x|S_\varepsilon^*)\otimes(A_\varepsilon\rho(x)A_\varepsilon^*)\\&=(|S_\varepsilon\delta_x\rangle\langle S_\varepsilon\delta_x|)\otimes(A_\varepsilon\rho(x)A_\varepsilon^*)\\&=(|\delta_{x-\varepsilon}\rangle\langle\delta_{x-\varepsilon}|)\otimes(A_\varepsilon\rho(x)A_\varepsilon^*).\end{aligned}$$

Further, using continuity of $\mathrm{L}_{(\mathcal{A})}$, we get

$$\begin{aligned}\mathrm{L}_{(\mathcal{U})}(\omega)&=\sum_{x\in¢^d}\mathrm{L}_{(\mathcal{W})}(|\delta_x\rangle\langle\delta_x|\otimes\rho(x))\\&=\sum_{x\in¢^d}(|\delta_{x-\varepsilon}\rangle\langle\delta_{x-\varepsilon}|)\otimes(A_\varepsilon\rho(x)A_\varepsilon^*)\\&=\sum_{x\in¢^d}(|\delta_x\rangle\langle\delta_x|)\otimes(A_\varepsilon\rho(x)A_\varepsilon^*)\\&=\sum_{x\in¢^d}|\delta_x\rangle\langle\delta_x|\otimes\tilde{\rho}(x)\\&=\omega.\end{aligned}$$

It is clearly that $\mathrm{L}_{(\mathcal{U})}(\omega)\in\mathcal{D}_{\text{Nuc}}(l^2(¢)\otimes £^d)$.

Difinition 3.2. Let $\mathcal{U}=\{A_\varepsilon\,|\,\varepsilon\in\Lambda^d\}$ be a Λ^d-generalized coin operators system on the space $£^d$. Then d -dimensional intrinsic quantum walk generated by $\mathcal{U}$ is a quantum dynamics model with the following features

We takes $l^2(¢)\otimes £^d$ as state space of the walk and its states are represented by density operator in $l^2(¢)\otimes £^d$; The evolution of walk is governed by equation

$$\omega^{(n)}=\mathrm{L}^n_{(\mathcal{U})}\left(\omega^{(0)}\right),\quad n\ge 0, \tag{20}$$

where $\mathrm{L}_{(\mathcal{U})}$ is the d -dimensional intrinsic quantum channel generated by $\mathcal{U}$, $\mathcal{L}_{(\mathcal{U})}^{n}$ denotes its n-th composition ($\mathcal{L}_{(\mathcal{U})}^{0}$ is defined as the corresponding identity mapping), and $\omega^{(n)}$ is the state of the walk at time n, in particular $\omega^{(0)}$ is the initial state.

In this case, for each $n \geq 0$, the function $x \mapsto \mathrm{Tr}[\omega^{(n)}(x)]$ on $\mathbb{Z}^{d}$ is referred to as the probability distribution of the walk at time n. In addition, the spaces $l^{2}(\mathbb{Z}^{d})$ and $\mathbb{C}^{d}$ are respectively called the position space and the coin space of the intrinsic quantum walk.

It is easy to see that the intrinsic open quantum walk as defined above is fully determined by the quantum channel $\mathcal{L}_{(\mathcal{U})}$, whereby we call $\mathcal{L}_{(\mathcal{U})}$ as intrinsic quantum channel. As seen before, the definition of $\mathcal{L}_{(\mathcal{U})}$ does not depend on any ONB in the position space $l^{2}(\mathbb{Z}^{d})$, thus the intrinsic open quantum walk driven by $\mathcal{L}_{(\mathcal{U})}$ (i.e., the open quantum walk defined by the definition 3.2) does not depend on any ONB of state space, which fully demonstrates its intrinsic properties in high-dimensional space.

IV. Conclusion

As seen, in this paper, we extend an open quantum walk model on the one-dimensional integer lattice $\mathbb{Z}$ to the d-dimensional integer lattice $\mathbb{Z}^{d}$. We introduce an intrinsic open quantum walk model on the d-dimensional integer lattice $\mathbb{Z}^{d}$, which we call the high-dimensional intrinsic open quantum walk model.

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